

Partial resampling to approximate covering integer programs*

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Abstract

We consider column-sparse positive covering integer programs, which generalize set cover and which have attracted a long line of research developing (randomized) approximation algorithms. We develop a new rounding scheme based on the Partial Resampling variant of the Lovász Local Lemma developed by Harris & Srinivasan (2013). This achieves an approximation ratio of $1 + \frac{\ln(\Delta_1+1)}{a_{\min}} + O(\sqrt{\frac{\log(\Delta_1+1)}{a_{\min}}})$, where a_{\min} is the minimum covering constraint and Δ_1 is the maximum ℓ_1 -norm of any column of the covering matrix (whose entries are scaled to lie in $[0, 1]$). When there are additional constraints on the sizes of the variables, we show an approximation ratio of $1 + O(\frac{\log(\Delta_1+1)}{a_{\min}^\epsilon} + \sqrt{\frac{\log(\Delta_1+1)}{a_{\min}}})$ to satisfy these size constraints up to multiplicative factor $1 + \epsilon$, or an approximation of ratio of $\ln \Delta_0 + O(\sqrt{\log \Delta_0})$ to satisfy the size constraints exactly (where Δ_0 is the maximum number of non-zero entries in any column of the covering matrix). We also show nearly-matching inapproximability and integrality-gap lower bounds. These results improve asymptotically, in several different ways, over results shown by Srinivasan (2006) and Kolliopoulos & Young (2005).

We show also that our algorithm automatically handles multi-criteria programs, efficiently achieving approximation ratios which are essentially equivalent to the single-criterion case and which apply even when the number of criteria is large.

1 Introduction

We consider *positive covering integer programs* – or simply covering integer programs (CIPs) – defined as follows (with \mathbf{Z}_+ denoting the set of non-negative integers). We have solution variables $x_1, \dots, x_n \in \mathbf{Z}_+$, and for $k = 1, \dots, m$, a system of m *covering constraints* of the form:

$$\sum_i A_{ki} x_i \geq a_k$$

Here A_k is an n -long non-negative vector; by scaling, we can assume that $A_{ki} \in [0, 1]$ and $a_k \geq 1$. We can write this more compactly as $A_k \cdot x \geq a_k$. We may optionally have constraints on the size of the solution variables, namely, that we require $x_i \in \{0, 1, \dots, d_i\}$; these are referred to as the

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multiplicity constraints. Finally, we have some linear objective function, represented by a vector $C \in [0, \infty)^n$. Our goal is to minimize $C \cdot x$, subject to the multiplicity and covering constraints.

This generalizes the set-cover problem, which can be viewed as a special case in which $a_k = 1$, $A_{ki} \in \{0, 1\}$. Solving set cover or integer programs exactly is NP-hard [12], so a common strategy is to obtain a solution which is approximately optimal. There are at least three ways one may obtain an approximate solution, where OPT denotes the optimal solution-value for the given instance:

1. the solution x may violate the optimality constraint, that is, $C \cdot x > \text{OPT}$;
2. x may violate the multiplicity constraint: i.e., $x_i > d_i$ for some i ;
3. x may violate the covering constraints: i.e., $A_k \cdot x < a_k$ for some k .

These three criteria are in competition. For our purposes, we will demand that our solution x completely satisfies the covering constraints. We will seek to satisfy the multiplicity constraints and optimality constraint as closely as possible. Our emphasis will be on the optimality constraints, that is, we seek to ensure that

$$C \cdot x \leq \beta \times \text{OPT}$$

where $\beta \geq 1$ is “small”. The parameter β , in this context, is referred to as the *approximation ratio*. More precisely, we will derive a randomized algorithm with the goal of satisfying $\mathbf{E}[C \cdot x] \leq \beta \times \text{OPT}$, where the expectation is taken over our algorithm’s randomness.

Many approximation algorithms for set cover and its extensions give approximation ratios as a function of m , the total number of constraints: e.g., it is known that the greedy algorithm has approximation ratio $(1 - o(1)) \ln m$ [18]. We often prefer a *scale-free* approximation ratio, that does not depend on the problem size but only on its structural properties. Two cases that are of particular interest are when the matrix A is *row-sparse* (a bounded number of variables per constraint) or *column-sparse* (each variable appears in a bounded number of constraints). We will be concerned solely with the column-sparse setting in this paper. The row-sparse setting, which generalizes problems such as vertex cover, typically leads to very different types of algorithms than the column-sparse setting.

Two common parameters used to measure the column sparsity of such systems are the maximum l_0 and l_1 norms of the columns; that is,

$$\Delta_0 = \max_i \#k : A_{ki} > 0, \quad \Delta_1 = \max_i \sum_k A_{ki}$$

Since the entries of A are in $[0, 1]$, we have $\Delta_1 \leq \Delta_0$; it is also possible that $\Delta_1 \ll \Delta_0$.

Approximation algorithms for column-sparse CIPs typically yield approximation ratios which are a function of Δ_0 or Δ_1 , and possibly other problem parameters as well. These algorithms fall into two main classes. First, there are greedy algorithms: they start by setting $x = 0$, and then increment x_i where i is chosen in some way which “looks best” in a myopic way for the residual problem. These were first developed by [3] for set cover, and later analysis (see [7]) showed that they give essentially optimal approximation ratios for set cover. These were extended to CIP in [8] and [5], showing an approximation ratio of $1 + \ln \Delta_0$. These greedy algorithms are often powerful, but they

are somewhat rigid. In addition, the greedy algorithms do not yield “oblivious” approximation ratios — that is, the greedy algorithm can only operate with knowledge of the objective function.

An alternative, and often more flexible, class of approximation algorithms is based on *linear relaxation*. There are a number of possible linear relaxations, but the simplest is one which we refer to as the *basic LP*. This LP has the same covering constraints as the original CIP, but replaces the constraint $x_i \in \{0, 1, \dots, d_i\}$ with the weaker constraint $x_i \in [0, d_i]$. The set of feasible points to the basic LP is a polytope, and one can find its exact optimal fractional solution \hat{x} . As this is a relaxation, we have $C \cdot \hat{x} \leq \text{OPT}$. It thus suffices to turn the solution \hat{x} into a random integral solution x satisfying $\mathbf{E}[C \cdot x] \leq \beta(C \cdot \hat{x})$. We will also see some stronger LP formulations, such as the Knapsack-Cover (KC) inequalities.

Randomized rounding is often employed to transform solutions to the basic LP back to a feasible integral solution. The simplest scheme, first applied to this context by [17], is to simply draw x_i as *independent* Bernoulli($\alpha \hat{x}_i$), for some $\alpha > 1$. When this is used, simple analysis using Chernoff bounds shows that $A_k \cdot x \geq a_k$ simultaneously for all k when $\alpha \geq 1 + c_0(\frac{\log m}{a_k} + \sqrt{\frac{\log m}{a_k}})$, where $c_0 > 0$ is some absolute constant. Thus, the overall solution $C \cdot x$ is within a factor of $1 + O(\frac{\log m}{a_{\min}} + \sqrt{\frac{\log m}{a_{\min}}})$ from the optimum, where $a_{\min} = \min_k a_k \geq 1$. One noteworthy aspect here is that this randomized rounding does not depend on the specific objective function; in this sense, it is “oblivious”, yielding a good expected value for any objective function.

In [19], Srinivasan gave a scale-free method of randomized rounding (ignoring multiplicity constraints), based on the FKG inequality and some proof ideas behind the Lovász Local Lemma (LLL). This gave an approximation ratio of $1 + O(\frac{\log(\Delta_0+1)}{a_{\min}} + \sqrt{\frac{\log a_{\min}}{a_{\min}} + \frac{\log(\Delta_0+1)}{a_{\min}}})$. The rounding scheme, by itself, only gave a positive (exponentially small) probability of achieving the desired approximation ratio. The algorithm of [19] also included a polynomial-time derandomization using the method of conditional expectations; this derandomization however requires knowledge of the objective function.

The algorithm of Srinivasan can potentially cause a large violation in the multiplicity constraints. In [13], Kolliopoulos & Young considered how to modify the algorithm of [19] to respect the multiplicity constraints. They gave two algorithms, which offer different types of approximation ratios. The first algorithm takes parameter $\epsilon \in (0, 1]$, violates each multiplicity constraint “ $x_i \leq d_i$ ” to at most “ $x_i \leq \lceil (1 + \epsilon)d_i \rceil$ ”, and has approximation ratio of $O(1 + \frac{\log(\Delta_0+1)}{a_{\min} \cdot \epsilon^2})$. (We refer to this situation as *ϵ -respect multiplicity*.) The second algorithm meets the multiplicity constraints exactly and achieves approximation ratio $O(1 + \log \Delta_0)$.

1.1 Our contributions

In this paper, we give a new randomized rounding scheme, based on the partial resampling variant of the LLL developed in [10] and some proof ideas developed in [9] for the LLL applied to systems of correlated constraints. We show the following result:

Theorem 1.1. *Suppose we have fractional solution \hat{x} for the basic LP. Let $\gamma = \frac{\ln(\Delta_1+1)}{a_{\min}}$. Then our randomized algorithm yields a solution $x \in \mathbf{Z}_+^n$ satisfying the covering constraints with probability*

one, and with

$$\mathbf{E}[x_i] \leq \hat{x}_i(1 + \gamma + 4\sqrt{\gamma})$$

The expected running time of this rounding algorithm is $O(mn)$.

This automatically implies that $\mathbf{E}[C \cdot x] \leq \beta C \cdot \hat{x} \leq \beta \times \text{OPT}$ for $\beta = 1 + \gamma + 4\sqrt{\gamma}$. Our algorithm has several advantages over previous techniques.

1. We give approximation ratios in terms of Δ_1 , the maximum l_1 -norm of the columns of A . Such bounds are always stronger than those phrased in terms of the corresponding l_0 -norm.
2. When Δ_1 is small, our approximation ratio is asymptotically smaller than that of [19]. In particular, we avoid the $\sqrt{\frac{\log a_{\min}}{a_{\min}}}$ term in our approximation ratio.
3. When Δ_1 is large, then our approximation ratio is roughly γ ; this is asymptotically optimal (including having the correct coefficient), and improves on [19].
4. This algorithm is quite efficient, essentially as fast as reading in the matrix A .
5. The algorithm is oblivious to the objective function — although it achieves a good approximation factor for any objective C , the algorithm itself does not use C in any way.

We find it interesting that one can “boil down” the parameters Δ_1, a_{\min} into a single parameter γ , which seems to completely determine the behavior of our algorithm.

Our partial resampling algorithm in its simplest form could significantly violate the multiplicity constraints. By choosing slightly different parameters for our algorithm, we can ensure that the multiplicity constraints are nearly satisfied, at the cost of a worsened approximation ratio:

Theorem 1.2. *Suppose we have a fractional solution \hat{x} for the basic LP. Let $\gamma = \frac{\ln(\Delta_1+1)}{a_{\min}}$. For any given $\epsilon \in (0, 1]$, our algorithm yields a solution $x \in \mathbf{Z}_+^n$ satisfying the covering constraints with probability one, and with*

$$x_i \leq \lceil \hat{x}_i(1 + \epsilon) \rceil, \quad \mathbf{E}[x_i] \leq \hat{x}_i(1 + 4\sqrt{\gamma} + 4\gamma/\epsilon)$$

This is an asymptotic improvement over the approximation ratio of [13], in three different ways:

1. It depends on the ℓ_1 -norm of the columns, not the ℓ_0 norm;
2. When γ is large, it is smaller by a full factor of $1/\epsilon$;
3. When γ is small, it gives an approximation ratio which approaches 1, at a rate independent of ϵ .

The two previous approximation ratios are all given in terms of the basic LP. We also give an approximation algorithm based on the KC inequalities, which is a stronger linear relaxation than the basic LP. This gives a different type of asymptotic guarantee, which is phrased in terms of the optimal integral solution (not the optimal basic LP solution):

Theorem 1.3. *There is a randomized algorithm running in expected polynomial time, yielding a solution $x \in \mathbf{Z}_+^n$ which satisfies the covering constraints, multiplicity constraints, and has*

$$C \cdot x \leq (1 + \ln \Delta_0 + O(\sqrt{\log \Delta_0})) OPT$$

This improves over the the corresponding approximation ratio of [13], in that it achieves the optimal leading coefficient of $\ln \Delta_0$.

There are many ways of parametrizing CIP's; we have chosen to focus on the parameters such as the minimum RHS value a_{\min} , the maximum ℓ_1 -column norm Δ_1 , and most importantly the ratio $\ln(\Delta_1 + 1)/a_{\min}$. Our approximation ratios are functions of these parameters; we show a number of matching lower bounds, which demonstrate that one cannot obtain significantly improved approximation ratios *which are parametrized as functions of these same parameters*. The formal statements of these results contain numerous qualifiers and technical conditions, but we summarize these here.

1. When γ is large, then assuming the Exponential Time Hypothesis, any polynomial-time algorithm to solve the CIP (ignoring multiplicity constraints), whose approximation ratio is parametrized as a function of γ , must have approximation ratio $\gamma - O(\log \gamma)$.
2. When γ is large, then the integrality gap between the basic LP and integral solutions which ϵ -respect multiplicity, is of order $\Omega(\gamma/\epsilon)$.
3. When γ is small, then the integrality gap of the basic LP is $1 + \Omega(\gamma)$.

In this sense, our approximation algorithms are nearly optimal as functions of the parameters γ, ϵ . On the other hand, there are many alternate parameters which could be analyzed instead, and alternate approximation ratio guarantees given (which would be incomparable to ours.)

Finally, we give an extension to covering programs with multiple linear criteria. Specifically, we show that *even conditional on our solution x satisfying all the covering constraints*, not only do we have $\mathbf{E}[C_l \cdot x] \leq \beta C_l \cdot \hat{x}$ but that in fact the values of $C_l \cdot x$ are concentrated, roughly equivalent to the x_i being independently distributed as Bernoulli with probability $\beta \hat{x}_i$. Thus, for each l there is a very high probability that we have $C_l \cdot x \approx C_l \cdot \hat{x}$ and in particular there is a good probability that we have $C_l \cdot x \approx C_l \cdot \hat{x}$ simultaneously for all l .

Theorem 1.4 (Informal). *Suppose we are given a covering system with a fractional solution \hat{x} and with r objective functions C_1, \dots, C_r , whose entries are in $[0, 1]$ and such that $C_\ell \cdot \hat{x} \geq \Omega(\log r)$ for all $\ell = 1, \dots, r$. Let $\gamma = \frac{\ln(\Delta_1 + 1)}{a_{\min}}$. Then our solution x satisfies the covering constraints with probability one; with probability at least $1/2$,*

$$\forall \ell \quad C_\ell \cdot x \leq \beta(C_\ell \cdot \hat{x}) + O(\sqrt{\beta(C_\ell \cdot \hat{x}) \log r})$$

where $\beta = 1 + \gamma + 4\sqrt{\gamma}$. (A similar result is possible, if we also want to ensure that $x_i \leq \lceil \hat{x}_i(1 + \epsilon) \rceil$; then the approximation ratio is $1 + 4\sqrt{\gamma} + 4\gamma/\epsilon$.)

This significantly improves on [19], in terms of both the approximation ratio as well as the running time. Roughly speaking, the algorithm of [19] gave an approximation ratio of $O(1 + \frac{\log(1 + \Delta_0)}{a_{\min}})$ (worse than the approximation ratio in the single-criterion setting) and a running time of $n^{O(\log r)}$ (polynomial time only when r , the number of objective functions, is constant).

1.2 Outline

In Section 2, we develop a randomized rounding algorithm when the fractional solution satisfies $\hat{x} \in [0, 1/\alpha]^n$; here $\alpha \geq 1$ is a key parameter which we will discuss how to select in later sections. This randomized rounding produces a binary solution vector $x \in \{0, 1\}^n$, for which $\mathbf{E}[x_i] \approx \alpha \hat{x}_i$.

In Section 3, we will develop a deterministic quantization scheme to handle fractional solutions of arbitrary size, using the algorithm of Section 2 as a subroutine. We will show an upper bound on the sizes of the variables x_i in terms of the fractional \hat{x}_i . We will also show an upper bound on $\mathbf{E}[x_i]$, which we state in a generalized form without making reference to column-sparsity or other properties of the matrix A .

In Section 4, we consider the case in which we have a lower bound a_{\min} on the RHS constraint vectors a_k , as well as an upper bound Δ_1 on the ℓ_1 -norm of the columns of A . Based on these values, we set key parameters of the rounding algorithm, to obtain good approximation ratios as a function of a_{\min}, Δ_1 . These approximation ratios do not respect multiplicity constraints.

In Section 5, we extend these results to take into account the multiplicity constraints as well. We give two types of approximation algorithms: in the first, we ϵ -respect the the multiplicity constraints. In the second, we respect the multiplicity constraints exactly.

In Section 6, we construct a variety of lower bounds on achievable approximation ratios. These are based on integrality gaps as well as hardness results. These show that the approximation ratios developed in Section 4 are essentially optimal for most values of ϵ and $\ln(\Delta_1 + 1)/a_{\min}$, particularly when $\ln(\Delta_1) \gg a_{\min}$.

In Section 7, we show that our randomized rounding scheme obeys a negative correlation property, allowing us to show concentration bounds on the sizes of the objective functions $C_l \cdot x$. This significantly improves on the algorithm of [19]; we show asymptotically better approximation ratios in many regimes, and we also give a polynomial-time algorithm regardless of the number of objective functions.

1.3 Comparison with the Lovász Local Lemma

One type of rounding scheme that has been used for similar types of integer programs is based on the LLL; we contrast this with our approach taken here.

The LLL, first introduced in [6], is often used to show that a rare combinatorial structure can be randomly sampled from a probability space. In the basic form of randomized rounding, one must ensure that the probability of a “bad-event” (an undesirable configuration of a subset of the variables) — namely, that $A_k \cdot x < a_k$ — is on the order of $1/m$; this ensures that, with high probability, no bad events occur. This accounts for the term $\log m$ in the approximation ratio. The power of the LLL comes from the fact that the probability of a bad-event is not compared with the total number of events, but only with the number of events it affects. Thus, one may hope to show approximation ratios which are independent of m .

At a heuristic level, the LLL should be applicable to the CIP problem. We have a series of bad-events, one for each covering constraint. Furthermore, because of our assumption that the system is column-sparse, each variable only affects a limited number of these bad-events. Thus, it should be possible to use the LLL to obtain a scale-free approximation ratio.

There has been prior work applying the LLL to packing integer programs, such as [14]. One technical problem with the LLL is that it only depends on whether bad-events affect each other, not the degree to which they do so. Bad-events which are only slightly correlated are still considered as dependent by the LLL. Thus, a weakness of the LLL for integer programs with arbitrary coefficients (i.e. allowing $A_{ki} \in [0, 1]$), is that potentially all the entries of A_{ki} could be extremely small yet non-zero, causing every constraint to affect each other by a tiny amount. For this reason, typical applications of the LLL to column-sparse integer programs have been phrased in terms of the l_0 column norm Δ_0 . For packing problems with no constraint-violation allowed, good approximations parametrized by Δ_0 , but *not* in general by Δ_1 , are possible [1].

In [11], Harvey addressed this technical problem by applying a careful, multi-step quantization scheme with iterated applications of the LLL, to discrepancy problems with coefficient matrices where the ℓ_1 norm of each column *and each row* is “small”.

The LLL, in its classical form, only shows that there is a small probability of avoiding all the bad-events. Thus, it does not lead to efficient algorithms. In [15], Moser & Tardos solved this long-standing problem by introducing a resampling-based algorithm. This algorithm initially samples all random variables from the underlying probability space, and will continue resampling subsets of variables until no more bad-events occur. Most applications of the LLL, such as [11], would yield polynomial-time algorithms using this framework.

In the context of integer programming, the Moser-Tardos algorithm can be extended in ways which go beyond the LLL itself. In [10], Harris & Srinivasan described a variant of the Moser-Tardos algorithm based on “partial resampling”. In this scheme, when one encounters a bad-event, one only resamples a random subset of the variables (where the probability distribution on which variables to resample is carefully chosen). This was applied for “assignment-packing” integer programs with small constraint violation. These bounds, like those of [11], depend on Δ_1 .

It is possible to formulate the CIP problem in the LLL framework, and to view our algorithm as a variant of the Moser-Tardos algorithm. This would achieve qualitatively similar bounds, albeit with asymptotics which are noticeably worse than the ones we give here. In particular, using the LLL directly, one cannot achieve approximation factors of the form $1 + \gamma$ when $\gamma \rightarrow \infty$; one obtains instead an approximation ratio of $1 + c\gamma$ where c is some constant strictly larger than one. The case when $\gamma \rightarrow 0$ is more complicated and there the LLL-based approaches appear to be asymptotically weaker by super-constant factors.

The technical core of our algorithm is an adaptation of the partial resampling MT algorithm of [10] combined with a methodology of [9] to yield improved probabilistic guarantees for LLL systems with correlated constraints. These techniques can only be used when the original fractional solution has entries which are small (and hence can be interpreted as probabilities); we develop a novel preprocessing step to handle large fractional entries which giving good guarantees on the multiplicity constraints.

Because so many different problem-specific techniques and calculations are combined with a variety

of LLL techniques, it is cumbersome to derive our algorithm directly as a special case or corollary of results from [10]. For the most part, we will discuss our algorithm in a self-contained way, keeping the comparison with the LLL more as informal motivation than technical guide.

2 The RELAXATION algorithm

We first consider the case when all the values of \hat{x} are small; this turns out to be the critical case for this problem. In this case, we present an algorithm which we label *RELAXATION*. Initially, this algorithm draws each x_i as an independent Bernoulli trials with probability $p_i = \alpha \hat{x}_i$, for some parameter $\alpha > 1$. This will satisfy many of the covering constraints, but there will still be some left unsatisfied. We loop over all such constraint; whenever a constraint k is unsatisfied, we modify the solution as follows: for each variable i which has $x_i = 0$, we set x_i to be an independent Bernoulli random variable with probability $\sigma A_{ki} \alpha \hat{x}_i$. Here $\sigma \in [0, 1]$ is another parameter which we will also discuss how to select.

For the remainder of this Section 2, we assume throughout that $\sigma \in [0, 1]$ and $\alpha > 1$ are given parameters, and that in addition we have $\hat{x}_i < 1/\alpha$ for all $i \in [n]$. We assume also that \hat{x} satisfies the covering constraints, i.e. $A_k \cdot \hat{x} \geq a_k$ for all $k = 1, \dots, m$. These assumptions will not be stated explicitly in the remainder.

Algorithm 1 The RELAXATION algorithm

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1: function RELAXATION( $\hat{x}, A, a, \sigma, \alpha$ )
2:   for  $i$  from  $1, \dots, n$  do ▷ Initialization
3:      $x_i \sim \text{Bernoulli}(\alpha \hat{x}_i)$ 
4:   while  $A \cdot x \not\geq a$  do ▷ The covering constraints are not all satisfied
5:     Let  $k$  be minimal such that  $A_k \cdot x < a_k$ 
6:     for  $i$  from  $1, \dots, n$  do
7:       if  $x_i = 0$  then
8:          $x_i \sim \text{Bernoulli}(\sigma A_{ki} \alpha \hat{x}_i)$ 
9:   return  $x$ 

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Note that this algorithm only increments the variables. Hence, when a constraint k is satisfied, it will remain satisfied until the end of the algorithm.

Whenever we encounter an unsatisfied constraint k and draw new values for the variables (lines 6–8), we refer to this as *resampling* the constraint k . There is an alternative way of looking at the resampling procedure, which seems counterintuitive but will be crucial for our analysis. Instead of setting each variable $x_i = 1$ with probability $\sigma A_{ki} \alpha \hat{x}_i$, we instead select a subset $Y \subseteq [n]$, where each i currently satisfying $x_i = 0$ goes into Y independently with probability σA_{ki} . Then, for each variable $i \in Y$, we draw $x_i \sim \text{Bernoulli}(p_i)$, where $p_i = \alpha \hat{x}_i$. It is clear that this two-part sampling procedure is equivalent to the one-step procedure described in Algorithm 1. In this case, we say that Y is the *resampled set* for constraint k . If $i \in Y$ (for any constraint k) we say that variable i is *resampled*.

For every variable i , we either have $x_i = 1$ at the initial sampling, or x_i first becomes equal to one

during some resampling of a constraint k ; or $x_i = 0$ at the end of the algorithm. If $x_i = 1$ for the first time at the j^{th} resampling of constraint k , we say i turns at (k, j) . If $x_i = 1$ initially, we say that i turns at 0.

In the algorithm as we have described, the first step is to set the variables x_i as independent Bernoulli with probability p_i . Our analysis, following [10] and [9], is based on an inductive argument, in which we consider what occurs when x is set to some arbitrary value. If $A \cdot x \geq a$, then the algorithm is already finished. If not, there will be a series of modifications made to x until it terminates. Given any fixed value of x , we will show upper bounds on the probability of certain future events.

Lemma 2.1. *Let Z_1, \dots, Z_j be subsets of $[n]$. The probability that the first j resampled sets for constraint k are respectively Z_1, \dots, Z_j is at most $\prod_{l=1}^j f_k(Z_l)$, where we define*

$$f_k(Z) = (1 - \sigma)^{-a_k} \prod_{i \in [n]} (1 - A_{ki}\sigma) \prod_{i \in Z} \frac{(1 - p_i)A_{ki}\sigma}{1 - A_{ki}\sigma}$$

Proof. For any integer $T \geq 0$, any integer $j \geq 0$, any sets $Z_1, \dots, Z_j \subseteq [n]$ and any vector $v \in \{0, 1\}^n$, we define the following random process and the following event $\mathcal{E}(T, j, Z_1, \dots, Z_j, v)$: Suppose that instead of drawing $x \sim \text{Bernoulli}(\alpha \hat{x}_i)$ as in line 3 of RELAXATION, we set $x = v$, and we continue the remaining steps of the RELAXATION algorithm (lines 4–8) until done. We say that in this process event $\mathcal{E}(T, j, Z_1, \dots, Z_j, v)$ has occurred if:

1. There are at $< T$ total resamplings
2. There are at least j resamplings of constraint k
3. The first j resampled sets for constraint k are respectively Z_1, \dots, Z_j .

We claim now that for any Z_1, \dots, Z_j , and $v \in \{0, 1\}^n$, and any integer $T \geq 0$, we have

$$P(\mathcal{E}(T, j, Z_1, \dots, Z_j, v)) \leq \frac{\prod_{l=1}^j f_k(Z_l)}{\prod_{i \in Z_1 \cup \dots \cup Z_j} (1 - p_i)} \quad (1)$$

(Note that $p_i < 1$ by our assumption $x_i < 1/\alpha$, and so the RHS of (1) is always well-defined.)

We shall prove (1) by induction on T . For the base case ($T = 0$) this is trivially true, because $\mathcal{E}(T, j, Z_1, \dots, Z_j, v)$ is impossible (there must be at least 0 resamplings), and so the LHS of (1) is zero while the RHS is non-negative. We move on to the induction step.

If $Av \geq a$, then the RELAXATION algorithm performs no resamplings. Thus, if $j \geq 1$, then event $\mathcal{E}(T, j, Z_1, \dots, Z_j, v)$ is impossible and again (1) holds. On the other hand, if $j = 0$, then the RHS of (1) is equal to one, and again this holds vacuously. So we suppose $Av \not\geq a$; let k' be minimal such that $A_{k'}v < a_{k'}$. Then the first step of RELAXATION is to resample constraint k' .

We observe that if $v_i = 1$ for any $i \in Z_1 \cup \dots \cup Z_j$, then the event $\mathcal{E}(T, j, Z_1, \dots, Z_j, v)$ is impossible. The reason for this is that we only resample variables which are equal to zero; thus variable i can

never be resampled for the remainder of the RELAXATION algorithm. In particular, we will never have i in any resampled set. Thus, as $i \in Z_1 \cup \dots \cup Z_j$, it is impossible for Z_1, \dots, Z_j to eventually be the resampled sets for constraint k . So if $v_i = 1$ for any $i \in Z_1 \cup \dots \cup Z_j$ then (1) holds vacuously.

Let x' denote the value of the variables after the first resampling (x' is a random variable). Then we observe that the remaining steps of the RELAXATION algorithm are equivalent to what would have occurred if we had set $x = x'$ initially.

Now, suppose that $k' \neq k$. Then after the first resampling, the event $\mathcal{E}(T, j, Z_1, \dots, Z_j, v)$ becomes equivalent to the event $\mathcal{E}(T-1, j, Z_1, \dots, Z_j, x')$. Thus, in this case, we have

$$\begin{aligned} P(\mathcal{E}(T, j, Z_1, \dots, Z_j, v)) &= P(\mathcal{E}(T-1, j, Z_1, \dots, Z_j, x')) \\ &\leq \frac{\prod_{l=1}^j f_k(Z_l)}{\prod_{i \in Z_1 \cup \dots \cup Z_j} (1 - p_i)} \quad \text{induction hypothesis} \end{aligned}$$

and this shows the induction step as desired. (Note that here we are able to bound the probability of the event $\mathcal{E}(T-1, j, Z_1, \dots, Z_j, x')$, even though x' is a random variable instead of a fixed vector, because our induction hypothesis applies to *all* vectors $v \in \{0, 1\}^n$.)

Next, suppose that $k = k'$. In this case, we observe that the following are necessary events for $\mathcal{E}(T, j, Z_1, \dots, Z_j, v)$:

- (B1) $Y = Z_1$, where Y is the first resampled set for constraint $k' = k$.
- (B2) For any $i \in Z_1 \cap (Z_2 \cup \dots \cup Z_j)$, in the first resampling step (which includes variable i), we draw $x_i = 0$.
- (B3) $\mathcal{E}(T-1, j-1, Z_2, Z_3, \dots, Z_j, x')$

The condition (B2) follows from the observation, made earlier, that $\mathcal{E}(T-1, j-1, Z_2, Z_3, \dots, Z_j, x')$ is impossible if $x'_i = 1$ but $i \in Z_2 \cup \dots \cup Z_j$. Any such $i \in Z_1$ must be resampled (due to condition (B1)), and it must be resampled to become equal to zero.

Let us first bound the probability of the condition (B1). As we put each i into Y with probability $A_{ki}\sigma$ independently, the probability that all $i \in Z_1$ go into Y is $\prod_{i \in Z_1} A_{ki}\sigma$. By the same token, if $v_i = 0$, then i avoids going into Y with probability $1 - A_{ki}\sigma$. Therefore, the overall probability of selecting $Y = Z_1$ is given by

$$\begin{aligned} P(Y = Z_1) &= \prod_{i \in Z_1} A_{ki}\sigma \prod_{i \notin Z_1, v_i=0} (1 - A_{ki}\sigma) \\ &= \left(\prod_{i \in Z_1} A_{ki}\sigma \right) \left(\prod_{i \notin Z_1} (1 - A_{ki}\sigma) \right) \left(\prod_{v_i=1} (1 - A_{ki}\sigma)^{-1} \right) \quad (\text{as } v_i = 0 \text{ for all } i \in Z_1) \\ &= \prod_{i \in [n]} (1 - A_{ki}\sigma) \prod_{i \in Z_1} \frac{A_{ki}\sigma}{1 - A_{ki}\sigma} \prod_{i: v_i=1} (1 - A_{ki}\sigma)^{-1} \end{aligned}$$

By definition of k' , we have that $A_k v < a_k$. By Proposition A.1, we thus have:

$$\prod_{i: v_i=1} (1 - A_{ki}\sigma)^{-1} \leq (1 - \sigma)^{-a_k}$$

further implying:

$$P(Y = Z_1) \leq (1 - \sigma)^{-a_k} \prod_{i \in [n]} (1 - A_{ki}\sigma) \prod_{i \in Z_1} \frac{A_{ki}\sigma}{1 - A_{ki}\sigma} \quad (2)$$

Next, let us consider the probability of (B2), conditional on (B1). Each $i \in Y$ is drawn independently as Bernoulli- p_i ; thus, the total probability of event (B2), conditional on (B1), is at most $\prod_{i \in Z_1 \cap (Z_2 \cup \dots \cup Z_j)} (1 - p_i)$.

Finally, let us consider the probability of event (B3), conditional on (B1) and (B2). Observe that the event $\mathcal{E}(T - 1, j - 1, Z_2, Z_3, \dots, Z_j, x')$ is conditionally independent of events (B1) and (B2), given x' . By the law of total probability, we have

$$\begin{aligned} & P(\mathcal{E}(T - 1, j - 1, Z_2, Z_3, \dots, Z_j, x') \mid (B1), (B2)) \\ &= \sum_{v' \in \{0,1\}^n} P(\mathcal{E}(T - 1, j - 1, Z_2, Z_3, \dots, Z_j, v') P(x' = v')) \\ &\leq \sum_{v' \in \{0,1\}^n} \frac{\prod_{l=2}^j f_k(Z_l)}{\prod_{i \in Z_2 \cup \dots \cup Z_j} (1 - p_i)} P(x' = v') \quad \text{induction hypothesis} \\ &= \frac{\prod_{l=2}^j f_k(Z_l)}{\prod_{i \in Z_2 \cup \dots \cup Z_j} (1 - p_i)} \end{aligned}$$

Thus, as (B1), (B2), and (B3) are necessary conditions for $\mathcal{E}(T, j, Z_1, \dots, Z_j, v)$, we have

$$\begin{aligned} & P(\mathcal{E}(T, j, Z_1, \dots, Z_j, v)) \\ &\leq (1 - \sigma)^{-a_k} \prod_{i \in [n]} (1 - A_{ki}\sigma) \prod_{i \in Z_1} \frac{A_{ki}\sigma}{1 - A_{ki}\sigma} \prod_{i \in Z_1 \cap (Z_2 \cup \dots \cup Z_j)} (1 - p_i) \times \frac{\prod_{l=2}^j f_k(Z_l)}{\prod_{i \in Z_2 \cup \dots \cup Z_j} (1 - p_i)} \\ &= (1 - \sigma)^{-a_k} \prod_{i \in [n]} (1 - A_{ki}\sigma) \prod_{i \in Z_1} \frac{A_{ki}\sigma}{1 - A_{ki}\sigma} \prod_{i \in Z_1} (1 - p_i) \times \frac{\prod_{l=2}^j f_k(Z_l)}{\prod_{i \in Z_1 \cup \dots \cup Z_j} (1 - p_i)} \\ &= f_k(Z_1) \times \frac{\prod_{l=2}^j f_k(Z_l)}{\prod_{i \in Z_1 \cup \dots \cup Z_j} (1 - p_i)} \end{aligned}$$

and the induction claim again holds.

Thus, we have shown that (1) holds for any integer $T \geq 0$ and any Z_1, \dots, Z_j , and $v \in \{0,1\}^n$. Next, for any sets Z_1, \dots, Z_j and any $v \in \{0,1\}^n$, let us define the event $\mathcal{E}(j, Z_1, \dots, Z_j, v)$ to be the event that, if we start the RELAXATION algorithm with $x = v$, then the first j resampled sets for constraint k are respectively Z_1, \dots, Z_j ; we make no condition on the total number of resamplings. Observe that we have the increasing chain

$$\mathcal{E}(0, j, Z_1, \dots, Z_j, v) \subseteq \mathcal{E}(1, j, Z_1, \dots, Z_j, v) \subseteq \mathcal{E}(2, j, Z_1, \dots, Z_j, v) \subseteq \dots$$

and $\mathcal{E}(j, Z_1, \dots, Z_j, v) = \bigcup_{T=0}^{\infty} \mathcal{E}(T, j, Z_1, \dots, Z_j, v)$. By countable additivity of the probability measure, we have:

$$P(\mathcal{E}(j, Z_1, \dots, Z_j, v)) = \lim_{T \rightarrow \infty} P(\mathcal{E}(T, j, Z_1, \dots, Z_j, v)) \leq \lim_{T \rightarrow \infty} \frac{\prod_{l=1}^j f_k(Z_l)}{\prod_{i \in Z_1 \cup \dots \cup Z_j} (1 - p_i)} = \frac{\prod_{l=1}^j f_k(Z_l)}{\prod_{i \in Z_1 \cup \dots \cup Z_j} (1 - p_i)}$$

So far, we have computed the probability of having Z_1, \dots, Z_j be the first j resampled sets for constraint k , *given that x is fixed to an arbitrary initial value v* . We now can compute the probability that Z_1, \dots, Z_j are the first j resampled sets for constraint k given that x is drawn as independent Bernoulli- p_i .

In the first step of the RELAXATION algorithm, we claim that a necessary event for Z_1, \dots, Z_j to be the first j resampled sets is to have $x_i = 0$ for each $i \in Z_1 \cup \dots \cup Z_j$; the rationale for this is equivalent to that for (B2). This event has probability $\prod_{i \in Z_1 \cup \dots \cup Z_j} (1 - p_i)$. Subsequently the event $P(\mathcal{E}(j, Z_1, \dots, Z_j, x))$ must occur.

The probability of $\mathcal{E}(j, Z_1, \dots, Z_j, x)$, conditional on $x_i = 1$ for all $i \in Z_1 \cup \dots \cup Z_j$, is at most $\frac{\prod_{l=1}^j f_k(Z_l)}{\prod_{i \in Z_1 \cup \dots \cup Z_j} (1 - p_i)}$ (by a similar argument to that of computing the probability of (B3) conditional on (B1), (B2)). Thus, the *overall* probability that the first j resampled sets for constraint k are Z_1, \dots, Z_j is at most

$$P(Z_1, \dots, Z_j \text{ are first } k \text{ resampled sets}) \leq \prod_{i \in Z_1 \cup \dots \cup Z_j} (1 - p_i) \times \frac{\prod_{l=1}^j f_k(Z_l)}{\prod_{i \in Z_1 \cup \dots \cup Z_j} (1 - p_i)} = \prod_{l=1}^j f_k(Z_l)$$

as desired. \square

We next compute $\sum_{Z \subseteq [n]} f_k(Z)$; such sums will recur in our calculations.

Proposition 2.2. *Suppose $\alpha > \frac{-\ln(1-\sigma)}{\sigma}$. For any constraint k define*

$$s_k = (1 - \sigma)^{-a_k} e^{-\sigma \alpha A_k \cdot \hat{x}}$$

Then $\sum_{Z \subseteq [n]} f_k(Z) \leq s_k < 1$ for all $k = 1, \dots, m$.

Proof. We have

$$\begin{aligned} \sum_{Z \subseteq [n]} f_k(Z) &= \sum_{Z \subseteq [n]} (1 - \sigma)^{-a_k} \prod_{i \in [n]} (1 - A_{ki} \sigma) \prod_{i \in Z} \frac{(1 - p_i) A_{ki} \sigma}{1 - A_{ki} \sigma} \\ &= (1 - \sigma)^{-a_k} \prod_{i \in [n]} (1 - A_{ki} \sigma) \sum_{Z \subseteq [n]} \prod_{i \in Z} \frac{(1 - p_i) A_{ki} \sigma}{1 - A_{ki} \sigma} \\ &= (1 - \sigma)^{-a_k} \prod_{i \in [n]} (1 - A_{ki} \sigma) \prod_{i=1}^n \left(1 + \frac{(1 - p_i) A_{ki} \sigma}{1 - A_{ki} \sigma} \right) \\ &= (1 - \sigma)^{-a_k} \prod_{i \in [n]} (1 - A_{ki} p_i \sigma) \\ &\leq (1 - \sigma)^{-a_k} e^{-\sigma \sum_i A_{ki} p_i} = (1 - \sigma)^{-a_k} e^{-\sigma \alpha A_k \cdot \hat{x}} \end{aligned}$$

Also, noting that \hat{x} satisfies the covering constraints (i.e., $A_k \cdot \hat{x} \geq a_k$), we have that

$$s_k = (1 - \sigma)^{-a_k} e^{-\sigma \alpha A_k \cdot \hat{x}} < (1 - \sigma)^{-a_k} e^{-\sigma \alpha a_k \frac{-\ln(1-\sigma)}{\sigma}} = 1$$

\square

Proposition 2.3. *For any constraint k and any $i \in [n]$, we have*

$$\sum_{Z \subseteq [n], Z \ni i} f_k(Z) \leq s_k A_{ki} \sigma$$

Proof. We have:

$$\begin{aligned} \sum_{\substack{Z \subseteq [n] \\ Z \ni i}} f_k(Z) &= \sum_{\substack{Z \subseteq [n] \\ Z \ni i}} (1 - \sigma)^{-a_k} \prod_{l \in [n]} (1 - A_{kl} \sigma) \prod_{l \in Z} \frac{(1 - p_l) A_{kl} \sigma}{1 - A_{kl} \sigma} \\ &= (1 - \sigma)^{-a_k} \frac{(1 - p_i) A_{ki} \sigma}{1 - A_{ki} \sigma} \prod_{l \in [n] - \{i\}} (1 - A_{kl} \sigma) \sum_{Z \subseteq [n], Z \ni i} \prod_{l \in Z - \{i\}} \frac{(1 - p_l) A_{kl} \sigma}{1 - A_{kl} \sigma} \\ &= (1 - \sigma)^{-a_k} (1 - p_i) A_{ki} \sigma \prod_{l \in [n] - \{i\}} (1 - A_{kl} \sigma) \prod_{l \in [n] - \{i\}} \left(1 + \frac{(1 - p_l) A_{kl} \sigma}{1 - A_{kl} \sigma} \right) \\ &= (1 - \sigma)^{-a_k} (1 - p_i) A_{ki} \sigma \prod_{l \in [n] - \{i\}} (1 - A_{kl} p_l \sigma) \\ &= (1 - \sigma)^{-a_k} (1 - p_i) A_{ki} \sigma e^{\sigma A_{ki} p_i} e^{-\sigma \alpha (A_k \cdot \hat{x})} \\ &= s_k (1 - p_i) A_{ki} \sigma e^{\sigma A_{ki} p_i} \end{aligned}$$

Now note that $A_{ki} \leq 1, \sigma \leq 1$ and hence $(1 - p_i) e^{\sigma A_{ki} p_i} \leq 1$. The claimed bound then holds. \square

To gain some intuition about this expression s_k , note that if we set $\sigma = 1 - 1/\alpha$ (which is not necessarily the optimal choice for the overall algorithm), then we have

$$s_k = \alpha^{a_k} e^{-A_k \cdot \hat{x}(\alpha - 1)}$$

and this can be recognized as the Chernoff lower-tail bound. Namely, this is an upper bound on the probability that a sum of independent $[0, 1]$ -random variables, with mean $\alpha A_k \cdot \hat{x}$, will become as small as a_k . This makes sense: for example at the very first step of the algorithm (before any resamplings are performed), then $A_k \cdot x$ is precisely a sum of independent Bernoulli variables with mean $\alpha A_k \cdot \hat{x}$. The event we are measuring (the probability that a constraint k is resampled) is precisely the event that this sum is smaller than a_k .

We next bound the running time of the algorithm.

Proposition 2.4. *Suppose $\alpha > \frac{-\ln(1-\sigma)}{\sigma}$. The expected number of resamplings steps made by the algorithm RELAXATION is at most $\sum_k \frac{1}{e^{\sigma \alpha A_k \cdot \hat{x}} (1-\sigma)^{a_k} - 1}$.*

Proof. Consider the probability that there are $\geq l$ resamplings of constraint k . A necessary condition for this to occur is that there are sets Z_1, \dots, Z_l such that Z_1, \dots, Z_l are respectively the first

l resampled sets for constraint k . Taking a union-bound over Z_1, \dots, Z_l , we have:

$$\begin{aligned}
P(\geq l \text{ resamplings of constraint } k) &\leq \sum_{Z_1, \dots, Z_l \subseteq [n]} P(Z_1, \dots, Z_l \text{ are first resampled sets for constraint } k) \\
&\leq \sum_{Z_1, \dots, Z_l \subseteq [n]} f_k(Z_1) \dots f_k(Z_l) \quad (\text{by Lemma 2.1}) \\
&= \left(\sum_{Z \subseteq [n]} f_k(Z) \right)^l \\
&\leq s_k^l \quad (\text{by Proposition 2.2})
\end{aligned}$$

As $s_k < 1$, this implies that the expected number of resamplings of constraint k is at most

$$\sum_{l=1}^{\infty} s_k^l = \frac{1}{1/s_k - 1} = \frac{1}{(1 - \sigma)^{a_k} e^{-\sigma \alpha A_k \cdot \hat{x}}}$$

□

We also give a crucial bound on the distribution of the variables x_i at the *end* of the resampling process.

Theorem 2.5. *Suppose $A_k \cdot \hat{x} \geq a_k$ for all $k = 1, \dots, m$ and $\alpha > \frac{-\ln(1-\sigma)}{\sigma}$. Then for any $i \in [n]$, the probability that $x_i = 1$ at the conclusion of RELAXATION algorithm is at most*

$$P(x_i = 1) \leq \alpha \hat{x}_i \left(1 + \sigma \sum_k \frac{A_{ki}}{e^{\sigma \alpha A_k \cdot \hat{x}} (1 - \sigma)^{a_k} - 1} \right)$$

Proof. There are two possible ways to have $x_i = 1$: either i turns at 0 or it turns at (k, j) for some $k \in [m]$, $j \geq 1$. The former event has probability p_i .

Suppose that i turns at (k, j) . In this case, there must be sets Z_1, \dots, Z_j such that:

(C1) The first j resampled sets for constraint k are respectively Z_1, \dots, Z_j

(C2) $i \in Z_j$

(C3) During the j th resampling of constraint k , we set $x_i = 1$.

Now, observe that the probability of (C3), conditional on (C1), (C2), is p_i . The reason for this is that event (C3) occurs *after* (C1), (C2) are already determined. Thus, we can use time-stochasticity to compute the conditional probability.

For any fixed $k \in [m]$ and any fixed sets Z_1, \dots, Z_j , the probability that Z_1, \dots, Z_j are the first j resampled sets is at most $f_k(Z_1) \dots f_k(Z_j)$ by Lemma 2.1

Thus, in total, the probability that events (C1)–(C3) hold for a fixed Z_1, \dots, Z_j where $i \in Z_j$, is at most $p_i f_k(Z_1) \dots f_k(Z_j)$.

We now take a union bound over all $k \in [m]$ and all integers $j \geq 1$ and all sets $Z_1, \dots, Z_j \subseteq [n]$ with $i \in Z_j$. This gives:

$$\begin{aligned}
P(x_i = 1) &\leq p_i + \sum_{k=1}^m \sum_{j=1}^{\infty} \sum_{\substack{Z_1, \dots, Z_j \subseteq [n] \\ i \in Z_j}} p_i f_k(Z_1) \dots f_k(Z_j) \\
&= p_i \left(1 + \sum_{k=1}^m \sum_{Z \subseteq [n], Z \ni i} f_k(Z) \sum_{j=1}^{\infty} \sum_{\substack{Z_1, \dots, Z_j \subseteq [n] \\ Z_j = Z}} f_k(Z_1) \dots f_k(Z_{j-1}) \right) \\
&= p_i \left(1 + \sum_{k=1}^m \sum_{Z \subseteq [n], Z \ni i} f_k(Z) \sum_{j=1}^{\infty} \sum_{Z_1, \dots, Z_{j-1} \subseteq [n]} f_k(Z_1) \dots f_k(Z_{j-1}) \right) \\
&= p_i \left(1 + \sum_{k=1}^m \sum_{Z \subseteq [n], Z \ni i} f_k(Z) \sum_{j=1}^{\infty} s_k^{j-1} \right) \quad \text{by Proposition 2.2} \\
&= p_i \left(1 + \sum_{k=1}^m \frac{\sum_{Z \subseteq [n], Z \ni i} f_k(Z)}{1 - s_k} \right) \quad \text{as } s_k < 1 \\
&\leq p_i \left(1 + \sum_{k=1}^m \frac{s_k A_{ki} \sigma}{1 - s_k} \right) \quad \text{(by Proposition 2.3)} \\
&= \alpha \hat{x}_i \left(1 + \sigma \sum_k \frac{A_{ki}}{e^{\sigma \alpha A_k \cdot \hat{x}} (1 - \sigma)^{a_k} - 1} \right)
\end{aligned}$$

□

3 Extension to the Case Where \hat{x}_i is Large

In the previous section, we described the RELAXATION algorithm under the assumption that $\hat{x}_i < 1/\alpha$ for all i . This assumption was necessary because each variable i is chosen to be drawn as a Bernoulli random variable with probability $p_i = \alpha \hat{x}_i$. In this section, we give a rounding scheme to cover fractional solutions \hat{x} of unbounded size. We first give an overview of this process.

Our goal is to extend the approximation ratio $\rho_i = \alpha \left(1 + \sigma \sum_k \frac{A_{ki}}{e^{\sigma \alpha A_k \cdot \hat{x}} (1 - \sigma)^{a_k} - 1} \right)$ of Section 2. First, note that if we have a variable i , and a solution to the LP with fractional value \hat{x}_i , we can sub-divide it into two new variables y_1, y_2 with fractional values \hat{y}_1, \hat{y}_2 such that $\hat{y}_1 + \hat{y}_2 = \hat{x}_i$. Now, whenever the variable x_i appears in the covering system, we replace it by $y_1 + y_2$. This process of sub-dividing variables can force all the entries in the fractional solution to be arbitrarily small. We can run the RELAXATION algorithm on this subdivided fractional solution, obtaining an integral solution y_1, y_2 and hence $x_i = y_1 + y_2$. Observe that the approximation ratios for the two new variables both equal to ρ_i itself. Thus $\mathbf{E}[x_i] = \mathbf{E}[y_1 + y_2] \leq \rho_i y_1 + \rho_i y_2 \leq \rho_i \hat{x}_i$.

By subdividing the fractional solution, we can always ensure that we obtain the same approximation for the general case (in which \hat{x} is unbounded) as in the case in which \hat{x} is restricted to entries of bounded size. However, this may violate the multiplicity constraints: in general, if we subdivide a fractional solution \hat{x}_i into $\hat{y}_1, \dots, \hat{y}_l$, and then set $x_i = y_1 + \dots + y_l$, then x_i could become as large as l .

There is another, simpler way to deal with large values \hat{x}_i : for any variable with $\hat{x}_i \geq 1/\alpha$, simply set $x_i = 1$. Then, we certainly are guaranteed that $\mathbf{E}[x_i] \leq \alpha \hat{x}_i \leq \rho_i \hat{x}_i$. Let us see what problems this procedure might cause. Consider some variable i with $\hat{x}_i = r \geq 1/\alpha$.¹ Because we have fixed $x_i = 1$, we may remove this variable from the covering system. When we do so, we obtain a residual problem A', a' , in which the i th column of A is replaced by zero and all the RHS vectors a_k are replaced by $a'_k = a_k - A_{ki}$.

Suppose that variable i appears in constraint k with another variable i' with $A_{ki} = 1$. We want to bound $\mathbf{E}[x'_i]$ in terms of $\beta_{i'}$; to do so, we want to show that constraint k contributes $\frac{A_{ki'}}{e^{\sigma \alpha A_k \cdot \hat{x}} (1-\sigma)^{a_k-1}}$ to $\mathbf{E}[x'_i]$. Now, in the residual problem, we replace a_k with $a_k - 1$ and we replace $A_k \cdot \hat{x}$ with $A_k \cdot \hat{x} - r$. Thus, constraint k contributes the following to $\beta'_{i'}$:

$$\text{Contribution} = \frac{A_{ki'}}{e^{\sigma \alpha (A_k \cdot \hat{x} - r)} (1-\sigma)^{a_k-1} - 1} = \frac{A_{ki'}}{e^{\sigma \alpha (A_k \cdot \hat{x})} (1-\sigma)^{a_k} e^{-\sigma \alpha r} (1-\sigma)^{-1} - 1}$$

Observe that if $r > \frac{-\ln(1-\sigma)}{\alpha \sigma}$, then this is *larger* than the original contribution term we wanted to show, namely $\rho_i = \frac{A_{ki'}}{e^{\sigma \alpha A_k \cdot \hat{x}} (1-\sigma)^{a_k}}$. Thus, there is a critical cut-off value $\theta = \frac{-\ln(1-\sigma)}{\alpha \sigma}$; when $\hat{x}_i > \theta$, then forcing $x_i = 1$ gives a good approximation ratio for variable i but may have a worse approximation ratio for other variables which interact with it.

We can now combine these two methods for handling large entries of \hat{x}_i . For any variable i , we first subdivide variable i into multiple variables $\hat{y}_1, \dots, \hat{y}_l$ with fractional value θ , along with one further entry $\hat{y}_{l+1} \in [0, \theta]$. We immediately set $y_1, \dots, y_l = 1$. If $\hat{y}_{l+1} \geq 1/\alpha$, we set $y_{l+1} = 1$ as well, otherwise we will apply the RELAXATION algorithm for it. At the end of this procedure, we know that $x_i = y_1 + \dots + y_{l+1} \leq (l+1) = \lceil \frac{\hat{x}_i}{\theta} \rceil$. We also know that $\mathbf{E}[x_i] \leq \alpha(\hat{y}_1 + \dots + \hat{y}_l) + \rho_i \hat{y}_{l+1} \leq \rho_i(\hat{y}_1 + \dots + \hat{y}_{l+1}) = \rho_i \hat{x}_i$. Thus, we get a good approximation ratio and a good bound on the multiplicity of x_i .

3.1 The ROUNDING algorithm

For each variable i , let $v_i = \lfloor \hat{x}_i / \theta \rfloor$, where we define

$$\theta = \frac{-\ln(1-\sigma)}{\alpha \sigma}$$

We define $F_i = \hat{x}_i - v_i \theta$ which we can write as $F_i = \hat{x}_i \bmod \theta$. We also define:

$$G_i = \begin{cases} 0 & \text{if } F_i < 1/\alpha \\ 1 & \text{if } F_i \geq 1/\alpha \end{cases}, \quad \hat{x}'_i = \begin{cases} F_i & \text{if } F_i < 1/\alpha \\ 0 & \text{if } F_i \geq 1/\alpha \end{cases}$$

We form the residual problem $a'_k = a_k - \sum_i A_{ki}(G_i + v_i)$. We then run the RELAXATION algorithm on the residual problem, which satisfies the condition that $x'_i \in [0, 1/\alpha]^n$. This is summarized in Algorithm 2.

¹To gain intuition, the reader may consider the case in which $r > 1$. In this case, it is obvious that this is a bad rounding procedure. It is instructive to trace through exactly why it fails badly.

Algorithm 2 The ROUNDING algorithm

```

1: function ROUNDING( $\hat{x}$ ,  $A$ ,  $\sigma$ ,  $\alpha$ )
2:   Set  $\theta = \frac{-\ln(1-\sigma)}{\alpha\sigma}$ .
3:   for  $i$  from  $1, \dots, n$  do
4:      $v_i = \lfloor \hat{x}_i / \theta \rfloor$ 
5:      $F_i = \hat{x}_i \bmod \theta$ .
6:      $G_i = 1$  if  $F_i \geq 1/\alpha$ ,  $G_i = 0$  otherwise.
7:      $\hat{x}'_i = 0$  if  $F_i \geq 1/\alpha$ ,  $\hat{x}'_i = F_i$  otherwise.
8:   for  $k$  from  $1, \dots, m$  do
9:     Set  $a'_k = a_k - \sum_i A_{ki}(G_i + v_i)$ 
10:  Compute  $x' = \text{RELAXATION}(\hat{x}', A, a', \sigma, \alpha)$ 
11:  Return  $x = G + v + x'$ 

```

We begin by showing a variety of simple bounds on the variables before and after the quantization steps.

Proposition 3.1. *Suppose that $x' \in \{0, 1\}^n$ satisfies the residual covering constraints, that is, $A_k \cdot x' \geq a'_k$ for all $k = 1, \dots, m$.*

Then the solution vector returned by the ROUNDING algorithm, defined by $x = G + v + x'$, satisfies the original covering constraints. Namely, $A_k \cdot x \geq a_k$ for all $k = 1, \dots, m$.

Proof. For each k we have:

$$A_k \cdot x = A_k \cdot (x' + G + v) = A_k \cdot x' + \sum_i A_{ki}(G_i + v_i) \geq a'_k + \sum_i A_{ki}(G_i + v_i) = a_k$$

□

Proposition 3.2. *For any $i \in [n]$ we have*

$$\hat{x}_i - v_i\theta - G_i\theta \leq \hat{x}'_i \leq \hat{x}_i - v_i\theta - G_i/\alpha$$

Proof. If $G_i = 0$, then both of the bounds hold with equality. So suppose $G_i = 1$.

In this case, we have $1/\alpha \leq \hat{x}_i - v_i\theta \leq \theta$. So $x_i - v_i\theta - G_i/\alpha \geq \theta - 1/\alpha \geq 0$ and $x_i - v_i\theta - G_i\theta \leq \theta - \theta = 0$ as required. □

Proposition 3.3. *For any i , at the end of the procedure ROUNDING, we have*

$$x_i \leq \left\lceil \hat{x}_i \frac{\alpha\sigma}{-\ln(1-\sigma)} \right\rceil$$

Proof. Note that $1/\theta = \frac{\alpha\sigma}{-\ln(1-\sigma)}$. So we must show that $x_i \leq \lceil \hat{x}_i / \theta \rceil$.

First, suppose \hat{x}_i is not a multiple of θ . Then $x_i = x'_i + G_i + \lfloor x_i / \theta \rfloor$. Note that if $G_i = 1$, then $\hat{x}'_i = 0$ which implies that $x'_i = 0$. So $G_i + v_i \leq 1$ and hence $x_i \leq 1 + \lfloor x_i / \theta \rfloor = \lceil x_i / \theta \rceil$.

Next, suppose \hat{x}_i is a multiple of θ . Then $G_i = \hat{x}'_i = 0$ and so $x'_i = 0$ and we have $x_i = \lfloor x_i/\theta \rfloor = \lceil x_i/\theta \rceil$. \square

The next result shows that the quantization steps can only *decrease* the inflation factor for the RELAXATION algorithm. Proposition 3.4 is the reason for our choice of θ .

Proposition 3.4. *For any constraint k , we have*

$$(1 - \sigma)^{a'_k} e^{\sigma \alpha A_k \cdot \hat{x}'} \geq (1 - \sigma)^{a_k} e^{\sigma \alpha A_k \cdot \hat{x}}$$

Proof. Let $r = \sum_i A_{ki}(G_i + v_i)$. By definition, we have $a'_k = a_k - r$. We also have:

$$\begin{aligned} A_k \cdot \hat{x}' &= \sum_i A_{ki} \hat{x}'_i \\ &\geq \sum_i A_{ki} (\hat{x}_i - v_i \theta - G_i \theta) \quad \text{by Proposition 3.2} \\ &= a_k - r \theta \end{aligned}$$

Then

$$\begin{aligned} (1 - \sigma)^{a'_k} e^{\sigma \alpha A_k \cdot \hat{x}'} &= (1 - \sigma)^{a_k - r} e^{\sigma \alpha A_k \cdot \hat{x}'} \\ &\geq (1 - \sigma)^{a_k - r} e^{\sigma \alpha (a_k - r \theta)} \\ &= (1 - \sigma)^{-a_k} e^{-\sigma \alpha a_k} \times ((1 - \sigma) e^{\theta \sigma \alpha})^{-r} \\ &= (1 - \sigma)^{-a_k} e^{-\sigma \alpha a_k} \end{aligned}$$

\square

We can now show an overall bound on the behavior of the ROUNDING algorithm

Theorem 3.5. *Suppose that $\alpha > \frac{-\ln(1-\sigma)}{\sigma}$. Then at the end of the ROUNDING algorithm, we have for each variable i*

$$\mathbf{E}[x_i] \leq \alpha \hat{x}_i \left(1 + \sigma \sum_k \frac{A_{ki}}{e^{\sigma \alpha a_k} (1 - \sigma)^{a_k} - 1} \right)$$

The expected number of resamplings for the RELAXATION algorithm is at most $\sum_k \frac{1}{e^{\sigma \alpha a_k} (1 - \sigma)^{a_k} - 1}$.

Proof. Define

$$T = 1 + \sigma \sum_k \frac{A_{ki}}{e^{\sigma \alpha a_k} (1 - \sigma)^{a_k} - 1}$$

By Theorem 2.5, the probability that $x'_i = 1$ is at most

$$\begin{aligned} P(x'_i = 1) &\leq \alpha \hat{x}'_i \left(1 + \sigma \sum_k \frac{A_{ki}}{(1 - \sigma)^{a'_k} e^{\sigma \alpha A_k \cdot \hat{x}'} - 1} \right) \\ &\leq \alpha \hat{x}'_i T \quad (\text{by Proposition 3.4}) \end{aligned}$$

So we estimate $\mathbf{E}[x_i]$ by:

$$\begin{aligned}
\mathbf{E}[x_i] &= v_i + G_i + \mathbf{E}[x'_i] \\
&\leq v_i + G_i + \alpha \hat{x}'_i T \\
&\leq v_i + G_i + \alpha(\hat{x}_i - \theta v_i - G_i/\alpha)T \quad (\text{by Proposition 3.2}) \\
&\leq v_i(1 - \alpha\theta) + \alpha \hat{x}_i T \\
&\leq \alpha \hat{x}_i T \quad \text{as } \alpha\theta = \frac{-\ln(1-\sigma)}{\sigma} \geq 1
\end{aligned}$$

This shows the bound on $\mathbf{E}[x_i]$. The bound on the expected number of resamplings is similar. \square

4 Bounds in terms of a_{\min}, Δ_1

So far, we have given bounds on the behavior of ROUNDING algorithm which are as general as possible. Theorem 3.5 can be applied to systems which have multiple types of variables and constraints. However, we can obtain a simpler bound by reducing these to two simple parameters, namely Δ_1 , the maximum ℓ_1 -norm of any column of A , and $a_{\min} = \min_k a_k$. We will first assume that $a_{\min} \geq 1, \Delta_1 \geq 1$. Later, Theorem 4.2 will show that we can always ensure that this holds with a simple pre-processing step.

Theorem 4.1. *Suppose we are given a covering system with $\Delta_1 \geq 1, a_{\min} \geq 1$ and with a fractional solution \hat{x} . Let $\gamma = \frac{\ln(\Delta_1+1)}{a_{\min}}$.*

Then with appropriate choices of σ, α we may run the ROUNDING algorithm on this system to obtain a solution $x \in \mathbf{Z}_+^n$ satisfying

$$\begin{aligned}
\mathbf{E}[x_i] &\leq \hat{x}_i(1 + \gamma + 4\sqrt{\gamma}) \\
x_i &\leq \left\lceil \hat{x}_i \frac{\frac{1}{2}\gamma + \sqrt{\gamma}}{\ln(1 + \sqrt{\gamma})} \right\rceil \quad \text{with probability one}
\end{aligned}$$

The expected running time of this algorithm is $O(mn)$.

Proof. We set $\sigma = 1 - 1/\alpha$, where $\alpha > 1$ is a parameter to be determined. Now note that we have $\frac{-\ln(1-\sigma)}{\sigma} = \alpha \frac{\ln \alpha}{\alpha-1} < \alpha$.

So we may apply Theorem 3.5; for each $i \in [n]$ we have:

$$\begin{aligned}
\mathbf{E}[x_i] &\leq \hat{x}_i \alpha \left(1 + \sigma \sum_k \frac{A_{ki}}{(1-\sigma)^{a_k} e^{\sigma \alpha a_k} - 1} \right) \\
&= \hat{x}_i \alpha \left(1 + (1 - 1/\alpha) \sum_k \frac{A_{ki}}{e^{a_k(\alpha-1)} \alpha^{-a_k} - 1} \right) \\
&\leq \hat{x}_i \alpha \left(1 + (1 - 1/\alpha) \sum_k \frac{A_{ki}}{e^{a_{\min}(\alpha-1)} \alpha^{-a_{\min}} - 1} \right) \\
&\leq \hat{x}_i \left(\alpha + (\alpha - 1) \frac{\Delta_1}{e^{a_{\min}(\alpha-1)} \alpha^{-a_{\min}} - 1} \right)
\end{aligned}$$

Now substituting $\alpha = 1 + \gamma + 2\sqrt{\gamma} > 1$ and $a_{\min} = \ln(\Delta_1 + 1)/\gamma$ gives

$$\mathbf{E}[x_i] \leq \hat{x}_i \left(1 + \gamma + 2\sqrt{\gamma} + \frac{(2\sqrt{\gamma} + \gamma)\Delta_1}{(\Delta_1 + 1)^{\frac{2\sqrt{\gamma} + \gamma - 2\ln(1 + \sqrt{\gamma})}{\gamma}} - 1} \right)$$

Proposition A.2 shows that this is decreasing function of Δ_1 . We are assuming $a_{\min} \geq 1$, which implies that $\Delta_1 \geq e^\gamma - 1$. We can thus obtain an upper bound by substituting $\Delta_1 = e^\gamma - 1$, yielding

$$\mathbf{E}[x_i] \leq \hat{x}_i \left(1 + \gamma + 2\sqrt{\gamma} + \frac{(e^\gamma - 1)(\gamma + 2\sqrt{\gamma})}{\frac{e^{\gamma + 2\sqrt{\gamma}}}{(\sqrt{\gamma} + 1)^2} - 1} \right) \quad (3)$$

Some simple analysis of the RHS of (3) shows that

$$\mathbf{E}[x_i] \leq \hat{x}_i(1 + \gamma + 4\sqrt{\gamma})$$

To show the bound on the size of x_i , we apply Proposition 3.3, giving

$$x_i \leq \left\lceil \hat{x}_i \frac{\alpha\sigma}{-\ln(1 - \sigma)} \right\rceil = \left\lceil \hat{x}_i \frac{\frac{1}{2}\gamma + \sqrt{\gamma}}{\ln(1 + \sqrt{\gamma})} \right\rceil$$

Next, let us analyze the runtime of this procedure. The initial steps of rounding and forming the residual can be done in time $O(mn)$. By Theorem 3.5, the expected number of resampling steps made by the RELAXATION algorithm is at most

$$\begin{aligned} \mathbf{E}[\text{Resampling Steps}] &\leq \sum_k \frac{1}{e^{a_k(\alpha-1)}\alpha^{-a_k} - 1} \\ &\leq \frac{m}{e^{a_{\min}(\alpha-1)}\alpha^{-a_{\min}} - 1} \\ &\leq \frac{m}{(\Delta_1 + 1)^{\frac{\gamma + 2\sqrt{\gamma} - 2\ln(1 + \sqrt{\gamma})}{\gamma}} - 1} \leq m \end{aligned}$$

In each resampling step, we must draw a new random value for all the variables; this can be easily done in time $O(n)$. \square

We now show how to ensure that $a_{\min} \geq \Delta_1 \geq 1$:

Theorem 4.2. *Suppose we are given a covering system A, a with $\gamma = \ln(\Delta_1 + 1)/a_{\min}$. Then, in time $O(mn)$, one can produce a modified system A', a' which satisfies the following properties:*

1. *The integral solutions of A, a are precisely the same as the integral solutions of A', a' ;*
2. *$a'_{\min} \geq 1$ and $\Delta'_1 \geq 1$;*
3. *We have $\gamma' \leq \gamma$, where $\gamma' = \ln(\Delta'_1 + 1)/a'_{\min}$.*

Proof. First, suppose that there is some entry A_{ki} with $A_{ki} > a_k$. In this case, set $A'_{ki} = a_k$. Observe that any integral solution to the constraint $A_k \cdot x \geq a_k$ also satisfies $A'_k \cdot x \geq a_k$, and vice-versa. This step can only decrease Δ_1 and hence $\gamma' \leq \gamma$.

After this step, one can assume that $A_{ki} \leq a_k$ for all k, i . Now suppose there are some constraints with $a_k \leq 1$. In this case, replace row A_k with $A'_k = A_k/a_k$ and replace a_k with $a'_k = 1$. Because of our assumption that $A_{ki} \leq a_k$ for all k, i , the new row of the matrix still satisfies $A'_k \in [0, 1]^n$. This step ensures that $a'_k \geq 1$ for all k . Also, every column in the matrix is scaled up by at most $1/a_k \leq 1/a_{\min}$, so we have $\Delta'_1 \leq \Delta_1/a_{\min}$ and $a'_{\min} = 1$. We then have

$$\gamma' = \ln(\Delta'_1 + 1)/a'_{\min} = \ln(\Delta_1/a_{\min} + 1) \leq \frac{\ln(\Delta_1 + 1)}{a_{\min}} = \gamma.$$

Finally, suppose that $\Delta_1 \leq 1$. In this case, observe that we must have $A_{ki} \leq \Delta_1$ for all k, i . Thus, we can scale up both A, a by $1/\Delta_1$ to obtain $A' = A/\Delta_1, a' = a/\Delta_1$. This gives $\Delta'_1 = 1, a'_{\min} = a_{\min}/\Delta_1$

$$\gamma' = \frac{\ln(1 + 1)}{a_{\min}/\Delta_1} \leq \frac{\ln(\Delta_1 + 1)}{a_{\min}} = \gamma$$

□

Corollary 4.3. *Let $\gamma = \frac{\ln(\Delta_1 + 1)}{a_{\min}}$. There is an algorithm running in expected polynomial time to obtain a solution $x \in \mathbf{Z}_+^n$ which satisfies the covering constraints and which satisfies*

$$C \cdot x \leq (1 + \gamma + O(\sqrt{\gamma})) OPT$$

where OPT is the optimal integral solution to the original CIP.

Proof. First, apply Theorem 4.2 to ensure that $\Delta_1 \geq 1, a_{\min} \geq 1$; the resulting CIP has a parameter $\gamma' = \frac{\ln(\Delta'_1 + 1)}{a'_{\min}} \leq \gamma$. Next, consider the corresponding basic LP, in which all multiplicity constraints are ignored. Let us denote this LP by \mathcal{Z} and let $z \in [0, \infty)^n$ be an optimal solution to it, of value $Z = C \cdot z$. Clearly $Z \leq OPT$ since \mathcal{Z} is a relaxation (in two separate ways — \mathcal{Z} ignores the integrality constraints as well as the multiplicity constraints.)

Now suppose we apply Theorem 4.1, and let us denote the solution we obtain (which is a random variable) by $x \in \mathbf{Z}_+^n$. This solution x satisfies $\mathbf{E}[C \cdot x] \leq (1 + \gamma' + 4\sqrt{\gamma'})Z \leq (1 + \gamma + 4\sqrt{\gamma})Z$. Also, since x satisfies all the covering constraints, then x is also a solution to the linear program \mathcal{Z} ; this implies that $C \cdot x \geq Z$ with probability one.

By applying Markov's inequality to the non-negative random variable $C \cdot x - Z$, we see that

$$P(C \cdot x \geq (1 + \gamma + 5\sqrt{\gamma})Z) \leq \frac{\gamma + 4\sqrt{\gamma}}{\gamma + 5\sqrt{\gamma}}$$

This is an increasing function of γ , and $\gamma \leq \ln(m + 1)$. Simple calculus shows that

$$P(C \cdot x \geq (1 + \gamma + 5\sqrt{\gamma})Z) \leq 1 - O\left(\frac{1}{\sqrt{\log m}}\right)$$

Thus, after repeating this process for $O(\sqrt{\log m})$ iterations (in expectation), we achieve an integral solution which satisfies the covering constraints and which satisfies $C \cdot x \leq (1 + \gamma + 5\sqrt{\gamma})Z \leq (1 + \gamma + O(\sqrt{\gamma}))OPT$. □

5 Respecting multiplicity constraints

Theorem 4.1 may considerably violate the multiplicity constraints. We will describe two algorithms to better satisfy these constraints: the first ensures the multiplicity constraints are approximately preserved, and gives an approximation ratio in terms of the basic LP. The second preserves the multiplicity constraints exactly, but gives an approximation factor only in terms of the ℓ_0 norm of the constraint matrix and the optimal integral solution.

Theorem 5.1. *Suppose we have a CIP with $\Delta_1 \geq 1$, $a_{\min} \geq 1$, and a solution \hat{x} to its basic LP. Let $\gamma = \frac{\ln(\Delta_1+1)}{a_{\min}}$.*

Let $\epsilon \in [0, 1]$ be given. Then, with an appropriate choice of σ, α we may run the ROUNDING algorithm to obtain a solution $x \in \mathbf{Z}_+^n$ satisfying

$$\begin{aligned} x_i &\leq \lceil \hat{x}_i(1 + \epsilon) \rceil && \text{with probability one} \\ \mathbf{E}[x_i] &\leq \hat{x}_i(1 + 4\sqrt{\gamma} + 4\gamma/\epsilon) \end{aligned}$$

The expected run-time is $O(mn)$.

Proof. First, suppose $\gamma \leq \epsilon^2/2$. In this case, apply Theorem 4.1; this ensures that $x_i \leq \lceil \hat{x}_i \frac{\gamma/2 + \sqrt{\gamma}}{\ln(1 + \sqrt{\gamma})} \rceil$ and some simple analysis shows that this is at most $\lceil \hat{x}_i(1 + \epsilon) \rceil$. We then have $\mathbf{E}[x_i] \leq 1 + 4\sqrt{\gamma} + \gamma \leq 1 + 4\sqrt{\gamma} + 4\gamma/\epsilon$ as desired.

Next, suppose $\gamma \geq \epsilon^2/2$. Set $\alpha = \frac{-(1+\epsilon)\ln(1-\sigma)}{\sigma}$, where $\sigma \in (0, 1)$ is a parameter to be determined. Then by Proposition 3.3, we have $x_i \leq \lceil \hat{x}_i(1 + \epsilon) \rceil$ at the end of the ROUNDING algorithm.

We clearly have $\alpha \geq \frac{-\ln(1-\sigma)}{\sigma}$ and so by Theorem 3.5:

$$\mathbf{E}[x_i] \leq \alpha \hat{x}_i \left(1 + \sigma \sum_k \frac{A_{ki}}{(1-\sigma)^{a_k} e^{\sigma \alpha a_k} - 1} \right) \leq \alpha \hat{x}_i \left(1 + \sigma \frac{\Delta_1}{(1-\sigma)^{-a_{\min}} - 1} \right)$$

Now set $\sigma = 1 - e^{-\gamma/\epsilon}$; observe that this is indeed in the range $(0, 1)$. This ensures that $(1 - \sigma)^{-a_{\min}} = \Delta_1 + 1$ and hence

$$\mathbf{E}[x_i] \leq \hat{x}_i \alpha (1 + \sigma) = \hat{x}_i \left(\epsilon^{-1} \left(2 + \frac{1}{e^{\gamma/\epsilon} - 1} \right) (1 + \epsilon) \gamma \right)$$

Simple calculus shows that $\epsilon^{-1} \left(2 + \frac{1}{e^{\gamma/\epsilon} - 1} \right) (1 + \epsilon) \gamma \leq 1 + \epsilon + (2 + 2/\epsilon) \gamma$. By our assumption that $\epsilon \in [0, 1]$ and our assumption that $\epsilon^2/2 \leq \gamma$, this is at most $1 + \sqrt{2\gamma} + 4\gamma/\epsilon$ as desired.

The bound on the running time follows the same lines as Theorem 4.1. \square

Next, we show how to exactly preserve multiplicity constraints. We follow here the approach of [13], which in turn builds on an approach of [2]: they construct a stronger linear program via the *knapsack-cover (KC)* inequalities. This LP has exponential size, but can be approximately optimized in polynomial time. We then round the resulting solution using Theorem 4.1. Although

this algorithm is discussed in great detail in [13] and [2], we give a self-contained presentation here to fill in a few technical details.

The key to the KC inequalities is to form a residual problem, given that a set of variables X is “pinned” to their maximal values.

Definition 5.2 (The pinned-residual problem). *Suppose we have a CIP problem with constraint matrix A , RHS vector a , and multiplicity constraints d . Given any $X \subseteq [n]$, we define the pinned-residual, denoted $PR(X)$, to be a new CIP problem A', a', d which we obtain as follows.*

1. For each $k = 0, \dots, m$, let $v_k = a_k - \sum_{i \in X} A_{ki} d_i$, and set

$$a'_k = \begin{cases} v_k & \text{if } v_k > 1 \\ 1 & \text{if } v_k \in (0, 1] \\ 0 & \text{if } v_k \leq 0 \end{cases}$$

2. For each k, i set:

$$A_{ki} = \begin{cases} 0 & \text{if } i \in X \\ 0 & \text{if } i \notin X, v_k \leq 0 \\ \min(1, \frac{A_{ki}}{v_k}) & \text{if } i \notin X, v_k \in (0, 1] \\ A_{ki} & \text{if } i \notin X, v_k > 1 \end{cases}$$

Observe that if any constraint has $a'_k = 0$, then it has effectively disappeared. Also, observe that for $i \in X$, the constraint matrix A' does not involve variable x_i (the column corresponding to i is zero). Hence, we may assume that any solution x to $PR(X)$ has $x_i = 0$ for $i \in X$.

Proposition 5.3 ([13],[2]). *For any $X \subseteq [n]$, the following hold:*

1. Any integral solution to the original CIP A, a, d also satisfies $PR(X)$.
2. $PR(X)$ has $a'_{\min} \geq 1, \Delta'_1 \leq \Delta_0$, where Δ_0 is the maximum ℓ_0 -column norm of A .

Theorem 5.4. *Given any CIP A, a, d, C , there is an algorithm which runs in expected polynomial time and returns a solution $x \in \mathbf{Z}_+^n$ satisfying the covering and multiplicity constraints, and satisfying*

$$Cx \leq (1 + \ln \Delta_0 + O(\sqrt{\log \Delta_0})) OPT,$$

where OPT is the optimal integral solution.

Proof. Let $\gamma_0 = \ln(\Delta_0 + 1)$ and let $\delta = \frac{\frac{1}{2}\gamma_0 + \sqrt{\gamma_0}}{\ln(1 + \sqrt{\gamma_0})}$.

We begin by finding a fractional solution \hat{x} which minimizes $C \cdot \hat{x}$, subject to the conditions that $\hat{x}_i \in [0, d_i]$ and such that \hat{x} satisfies $PR(\{i \mid \hat{x}_i \geq d_i/\delta\})$. This can be done using the ellipsoid method: given some putative \hat{x} , one can form $PR(\{i \mid \hat{x}_i \geq d_i/\delta\})$ and determine which constraint in it, if any, is violated. (See [13] for more details.)

Suppose we are given some optimal LP solution \hat{x} satisfying this condition. By Proposition 5.3, any optimal integral solution satisfies $\text{PR}(Y)$ for all $Y \subseteq [n]$, and in particular is a solution to the given LP. Thus, $C \cdot \hat{x} \leq \text{OPT}$.

Let $X = \{i \mid \hat{x}_i \geq d_i/\delta\}$. Set $x_i = d_i$ for $i \in X$. For $i \notin X$, we run the algorithm of Theorem 4.1 on $\text{PR}(X)$ to obtain a random solution x_i .

For $i \in X$, we clearly have $x_i \leq d_i$. Observe that by Proposition 5.3, $\text{PR}(X)$ has $\gamma' \leq \gamma_0$. So for $i \notin X$, we have $x_i \leq \lceil \delta \hat{x}_i \rceil$; this is at most $\lceil d_i \rceil = d_i$ by definition of X . So x satisfies the multiplicity constraints.

Next, for $i \in X$ we clearly have $\mathbf{E}[x_i] \leq d_i \leq \hat{x}_i \delta \leq x_i(\gamma_0 + O(1))$. Also, for $i \notin X$, we have

$$\mathbf{E}[x_i] \leq \hat{x}_i(1 + \gamma' + 4\sqrt{\gamma'});$$

by Proposition 5.3 this is $\leq \hat{x}_i(1 + \gamma_0 + 4\sqrt{\gamma_0})$.

Thus,

$$\mathbf{E}[C \cdot x] \leq (1 + \gamma_0 + c\sqrt{\gamma_0})\text{OPT}$$

On the other hand, since x satisfies the covering constraints and multiplicity constraints, we have $C \cdot x \geq \text{OPT}$ with probability one. By Markov's inequality applied to the non-negative random variable $C \cdot x - \text{OPT}$,

$$P(C \cdot x \geq (1 + \gamma_0 + 2c\sqrt{\gamma_0})\text{OPT}) \leq \frac{\gamma_0 + c\sqrt{\gamma_0}}{\gamma_0 + 2c\sqrt{\gamma_0}} \leq 1 - O\left(\frac{1}{\sqrt{\gamma_0}}\right) \leq 1 - O\left(\frac{1}{\sqrt{\log m}}\right).$$

Thus, after repeating this process for $O(\sqrt{\log m})$ iterations (in expectation), we achieve a solution satisfying

$$C \cdot x \leq (1 + \gamma_0 + 2c\sqrt{\gamma_0})\text{OPT} \leq (1 + \ln \Delta_0 + O(\sqrt{\log \Delta_0}))\text{OPT}$$

□

6 Lower bounds on approximation ratios

In this section, we provide lower bounds on the approximation ratios of CIP algorithms. These bounds fall into two categories, namely, inapproximability of CIP (which follows from inapproximability of set cover), and integrality gaps for the basic LP. The formal statements of these results contain numerous qualifiers and technical conditions. So, we will summarize our results informally here:

1. Under the hypothesis $P \neq NP$, then any polynomial-time algorithm to solve the CIP while ignoring multiplicity constraints, which gives an approximation ratio parametrized as a function of γ , cannot have an approximation ratio of the form $(1 - \alpha) \ln \gamma$, where $\alpha > 0$ is any constant.

2. Under the Exponential Time Hypothesis (ETH), then any polynomial-time algorithm to solve the CIP while ignoring multiplicity constraints, which gives an approximation ratio parametrized as a function of γ , cannot have an approximation ratio of the form $\ln \gamma - C \ln \ln \gamma$, where C is some specific universal constant.
3. When γ is large, then the gap between solutions to the basic LP, and integral solutions which ϵ -respect the multiplicity constraints, can be as large as $\Omega(\gamma/\epsilon)$. By contrast, the algorithm of Theorem 5.1 achieves approximation ratio $O(\gamma/\epsilon)$.
4. When γ is small, the integrality gap of the basic LP is $1 + \Omega(\gamma)$; by contrast, the algorithm of Theorem 4.1 achieves approximation ratio $1 + O(\sqrt{\gamma})$.

We note that one might wish to formulate an approximation ratio in terms of many possible parameters; two natural ones are Δ_1, a_{\min} but there are numerous others including Δ_0, n, m , etc. Lower bounds for the approximation ratio are very difficult to state in the context of these multiparametric approximations. Thus, it is quite possible that one is able to show approximation ratios which are incomparable to ours, perhaps even ones which are stronger for most natural problem instances. However, approximation ratios which are functions of $\gamma = \frac{\Delta_1}{a_{\min}}$ cannot be significantly improved. (It is possible that there is an alternate and stronger functional form, which depends on Δ_1, a_{\min} in a more complicated way than their ratio.)

6.1 Hardness results

Set Cover is a well-studied special case of CIP; many of the hardness results for Set Cover thus automatically imply corresponding hardness results for CIP. These hardness results are all based on a construction of Feige [7], which was later strengthened by Moshkovitz [16]. These results showed that assuming $P \neq NP$, then for any constant $\alpha > 0$, set cover on a domain of size n can be approximated within a factor of $(1 - \alpha) \ln n$ in polynomial time. Dinur & Steurer [4] showed that under the Exponential Time Hypothesis, then set cover cannot be approximated to within a factor of $\ln n - C \ln \ln n$, where C is a universal constant.

Proposition 6.1. *Suppose that there is a function $f : [0, \infty) \rightarrow [0, \infty)$ and a polynomial-time algorithm \mathcal{A} to approximate CIP, such that \mathcal{A} is guaranteed to achieve approximation ratio $f(\frac{\ln(\Delta_1+1)}{a_{\min}})$. Then:*

1. *Assuming $P \neq NP$, there cannot be any $\alpha > 0$ such that $f(x) \leq (1 - \alpha)x$ for all sufficiently large x .*
2. *Assuming ETH, one cannot have $f(x) \leq x - C \ln x$ for all sufficiently large x , where C is some universal constant.*

Similarly, suppose that there is a function $f : [0, \infty) \rightarrow [0, \infty)$ and a polynomial-time algorithm \mathcal{A} to approximate CIP, such that \mathcal{A} is guaranteed to achieve approximation ratio $f(\ln(\Delta_0+1))$. Then:

1. *Assuming $P \neq NP$, there cannot be any $\alpha > 0$ such that $f(x) \leq (1 - \alpha)x$ for all sufficiently large x .*

2. Assuming *ETH*, one cannot have $f(x) \leq x - C \ln x$ for all sufficiently large x , where C is some universal constant.

Proof. Set Cover instances on a domain of size n can be encoded as CIP instances with $\Delta_0 = \Delta_1 \leq n - 1$ and $a_{\min} = 1$. Namely, for each item $i \in [n]$, we construct a constraint $\sum_{j|i \in S_j} x_j \geq 1$, where x_j is an indicator variable for the set S_j appearing in the cover. The ℓ_0 and ℓ_1 -column norms corresponding to a variable x_j are both $|S_j|$. We may assume that none of the sets S_j is equal to $[n]$, and so Δ_1, Δ_0 are both at most $n - 1$. \square

In particular, for γ (respectively Δ_0) large, the approximation ratio guarantees of Theorem 4.1 (respectively Theorem 5.4), are optimal up to first-order.

6.2 Integrality gaps for the regime in which $\gamma \rightarrow \infty$

We next show a variety of integrality gaps for the basic LP. These constructions work as follows: we give a CIP instance, as well as an upper bound on the weight of the fractional solution \hat{T} for the basic LP and a lower bound on the weight of any integral solution T . This implies that algorithm which start from the basic LP solution must cause the weight to increase by at least T/\hat{T} .

In this section, we show integrality gaps matching Theorems 4.1 and 5.1 when γ is large.

Proposition 6.2. *Let $a \geq 1, m \geq 1$ be given. There is a CIP program with m covering constraints and no multiplicity constraints which satisfies the following properties:*

1. *All the RHS values are equal to a common value a .*
2. *The entries of the constraint matrix A are in $\{0, 1\}$.*
3. *Let \hat{T} be the optimal value of this basic LP, and let T be the optimal value of the covering program itself.*

$$T/\hat{T} \geq \frac{\ln m}{a} - c \frac{\log \log m}{a} \geq \gamma - c \log \gamma$$

for some universal constant $c \geq 0$.

Proof. First, we claim that we can assume that m is larger than any desired constant. For, suppose $m \leq m_0$. Then, for some constant $c > 0$, we have $\ln m - c \log \log m \leq 1$ for all $m \leq m_0$. We certainly have $T/\hat{T} \geq 1$, so we have $T/\hat{T} \geq \ln m - c \log \log m \geq \frac{\ln m - c \log \log m}{a}$. Likewise, we can assume that $\ln m \geq a$. We will make both of these simplifications for the remainder of the proof.

We will form the m constraints randomly as follows: we select exactly s positions i_1, \dots, i_s uniformly at random in $[s]$ without replacement, where $s = \lceil pn \rceil$; here $n \rightarrow \infty$ and $p \rightarrow 0$ as functions of m . We then set $A_{ki_1} = \dots = A_{ki_s} = 1$; all other entries of A_k are set to zero. The RHS vector is always equal to a . The objective function C is defined by $C \cdot x = \sum x_i$; that is, each variable is assigned weight one.

We can form a fractional solution \hat{x} by setting $\hat{x}_i = a/s$. As each constraint contains exactly s entries with coefficient one, this satisfies all the covering constraints. Thus, the optimal fractional solution has value $\hat{T} \leq na/s = a/p$.

Now suppose we fix some integral solution of weight $\sum x_i = t$. Let $I \subseteq [n]$ denote the support of x , that is, the values $i \in [n]$ such that $x_i > 0$; we have $|I| = r \leq t$. In each constraint k , there is a probability of $\binom{n-r}{s}/\binom{n}{s}$ that $A_{ki} = 0$ for all $i \in I$. If this occurs, then certainly $A \cdot x = 0$ and the covering constraint is violated. Thus, the probability that x satisfies constraint k is at most $1 - \frac{\binom{n-r}{s}}{\binom{n}{s}}$. As all the constraints are independent, the total probability that x satisfies all m constraints is at most:

$$\begin{aligned} P(x \text{ satisfies all constraints and has weight } t) &\leq \left(1 - \frac{\binom{n-r}{s}}{\binom{n}{s}}\right)^m \\ &\leq \exp\left(-m \frac{\binom{n-r}{s}}{\binom{n}{s}}\right) \\ &\leq \exp\left(-m \left(\frac{n-s-(t-1)}{n}\right)^t\right) \\ &\leq \exp(-m(1-p-t/n)^t) \quad \text{as } s \leq pn+1 \end{aligned}$$

We want to ensure that there are *no* good integral solutions. To upper-bound the probability that there exists such a good x , we take a union-bound over all integral x . In fact, our estimate only depended on specifying the support of x , not the values it takes on there, so we only need to take a union bound over all subsets of $[n]$ of cardinality $\leq t$. There are at most $\sum_{r=0}^t \binom{n}{r} \leq n^t$ such sets, and thus we have

$$\begin{aligned} P(\text{Some } x \text{ satisfies all constraints}) &\leq n^t \exp(-m(1-p-t/n)^t) \\ &\leq \exp(t \ln n - m(1-p)^t + mt^2/n) \end{aligned}$$

We now set $n = mt$, and obtain

$$\begin{aligned} P(\text{Some } x \text{ satisfies all constraints}) &\leq \exp(t(1 + \ln(mt)) - m(1-p)^t) \\ &\leq \exp(t^2 \ln m - m \exp(-pt - p^2 t)) \quad \text{for } m, p, t \text{ sufficiently small} \end{aligned}$$

If this expression is smaller than one, then that implies that there is a positive probability that no integral solution exists. Hence, we can ensure that all integral solutions satisfy $T > t$. Now, some simple analysis shows that this expression is < 1 when $p = 1/\ln m$ and $t = p^{-1}(\ln m - 10 \ln \ln m)$ and m sufficiently large. Thus we have

$$\begin{aligned} T/\hat{T} &\geq \frac{p^{-1}(\ln m - 10 \ln \ln m)}{a/p} \\ &\geq \frac{\ln m - O(\log \log m)}{a} \\ &\geq \frac{\ln(D+1)}{a} - O\left(\log\left(\frac{\log(D+1)}{a}\right)\right) \end{aligned}$$

as we have claimed. □

This argument can be adjusted to take into account a $(1+\epsilon)$ violation of the multiplicity constraints.

Proposition 6.3. *Let a, m be given integer parameters and let $\epsilon \in (0, 1)$. Then there is a CIP instance on m constraints which share a common RHS value a and a parameter $d \geq 0$ such that the fractional solution $\hat{x} \in [0, d]^n$ has objective value \hat{T} , the optimal integral solution in $x \in \{0, 1, \dots, \lceil (1+\epsilon)d \rceil\}^n$ has objective value T , and*

$$\hat{T}/T \geq \frac{\ln m - c \ln \ln m}{a\epsilon}$$

for some universal constant $c \geq 0$.

Hence, the basic LP solution cannot be rounded to within $o(\frac{\gamma}{\epsilon})$ while ϵ -respecting multiplicity.

Proof. Let A be the CIP instance constructed of Proposition 6.2 in n variables and m constraints and with RHS value equal to one. By construction, it satisfies $T/\hat{T} \geq \ln m - c \ln \ln m$ for some constant $c \geq 0$.

We form a new CIP instance A' on $n+m$ variables and m constraints; for each constraint $k = 1, \dots, m$ we set

$$\frac{a}{K(1+\epsilon)+1}x_{m+k} + \sum_{i=1}^n A_{ki}x_i \geq a$$

and we have an objective function $C \cdot x = \sum_{i=1}^n x_i$; that is, each variable x_1, \dots, x_n has weight one, and each variable x_{n+1}, \dots, x_{n+m} has weight zero. We set $d_i = \infty$ for $i = 1, \dots, n$ and we set $d_i = K$ for $i = m+1, \dots, m+n$; here K is a large integer parameter, which we will specify shortly. (In particular, for K sufficiently large, all the coefficients in this constraint are in the range $[0, 1]$.)

The resulting CIP instance contains m constraints and $a_{\min} = a$. Now suppose that \hat{x} is a fractional solution to the original CIP instance. Then let $v = \frac{a(1+\epsilon K)}{1+(1+\epsilon)K}$ and consider the fractional solution \hat{x}' defined by

$$\hat{x}'_i = \begin{cases} v\hat{x}_i & \text{if } i \leq n \\ K & \text{if } n+1 \leq i \leq n+m \end{cases}$$

Observe that for any constraint k we have

$$\begin{aligned} \frac{a}{K(1+\epsilon)+1}\hat{x}_{m+k} + \sum_{i=1}^n A_{ki}\hat{x}'_i &= \frac{a}{K(1+\epsilon)+1}K + v \sum_{i=1}^n A_{ki}\hat{x}_i \\ &\geq \frac{a}{K(1+\epsilon)+1}K + v \quad \text{as } A_k \cdot \hat{x} \geq a_k = 1 \\ &= a \end{aligned}$$

and so this is a valid LP solution. Thus the fractional objective value is at most $\hat{T}' \leq \sum_{i=1}^n v\hat{x}'_i = v\hat{T}$.

On the other hand, consider an integral solution x' . As $x_{m+k} \leq \lceil (1+\epsilon)K \rceil$, we have that for all $k \in [m]$:

$$\frac{a}{K(1+\epsilon)+1}(1+\epsilon)K + \sum_{i=1}^n A_{ki}x_i \geq a$$

which implies that $\sum_{i=1}^n A_{ki}x_i > 0$.

As all the entries of A_{ki} are in $\{0, 1\}$, this implies that $\sum_{i=1}^n A_{ki}x_i \geq 1$, and so x is an integral solution to the original CIP instance. Thus, its objective value is at least $T' \geq T$, where T is the optimal integral solution to the original A .

Thus we have that

$$\frac{T'}{\hat{T}'} \geq \frac{T}{v\hat{T}} \geq \frac{\ln m - c \ln \ln m}{v}$$

Taking the limit as $K \rightarrow \infty$, we see that for any $\delta > 0$ there exists a CIP with integrality gap

$$\frac{T'}{\hat{T}'} \geq \frac{(1 + \epsilon)(\ln m - c \ln \ln m)}{a\epsilon} - \delta$$

In particular, as $\epsilon > 0$, we can select δ sufficiently small so that

$$\frac{T'}{\hat{T}'} \geq \frac{\ln m - c \ln \ln m}{a\epsilon}$$

□

In light of this result, we note that Theorem 5.1 has an optimal approximation ratio in terms of γ, ϵ for $\gamma \rightarrow \infty$, up to a constant factor. However, this integrality gap construction does not apply to Theorem 5.4, which uses a stronger LP formulation (the KC constraints). For this reason, Theorem 5.4 is able to achieve an approximation ratio which remains bounded as $\epsilon \rightarrow 0$.

6.3 Integrality gaps for the regime $\gamma \rightarrow 0$

We next show an integrality gap for the case of small γ . To our knowledge, this is the first non-trivial hardness result in this regime; previous works show, for instance, integrality gaps of the form $\Omega(\gamma)$, which is of course vacuous when $\gamma \approx 0$.

This integrality gap does not match Theorem 4.1 precisely; here, we obtain an integrality gap of order $1 + \Omega(\gamma)$ while Theorem 4.1 gives the weaker approximation ratio $1 + O(\gamma)$. Our construction here is based on an integrality gap of [20] for set cover, which we extend to CIP by allowing large RHS values.

Proposition 6.4. *For any $g \in (0, 1)$ and $m \geq 2^{1+14/g}$, there is a CIP with m covering constraints and no multiplicity constraints, all the RHS values equal to a common value a where $\frac{\ln m}{a} \leq g$, and which satisfies also the following integrality gap property: Let \hat{T} be the optimal value of the basic LP and let T be the optimal integral value. Then*

$$T/\hat{T} \geq 1 + g/8 \geq 1 + \Omega(\gamma)$$

In particular, it is impossible to guarantee an approximation ratio of the form $1 + o(\frac{\log m}{a_{\min}})$ as a function of the basic LP solution.

Proof. We set $n = 2^q - 1$ where $q = \lfloor \log_2 m \rfloor$. We will view the integers from $1, \dots, n$ as corresponding to the non-zero binary strings of length q . Thus, if $i, i' \in \{1, \dots, n\}$, then we write $i \cdot i'$ to denote the binary dot-product. Namely if we have $i = i_0 + 2i_1 + 4i_2 + \dots$ and $i' = i'_0 + 2i'_1 + 4i'_2 + \dots$ where $i_j, i'_j \in \{0, 1\}$, then we define $i \cdot i' = \bigoplus_{l=0}^{q-1} i_l i'_l$.

The covering system is defined as follows: For each $k \in \{1, \dots, n\}$ we have a constraint

$$\sum_{i: (k \cdot i)=0} x_i \geq a \quad \text{where } a = \frac{q-1}{g}$$

The objective function is $C \cdot x = \sum_{i=1}^n x_i$. This has $n \leq m$ constraints. Observe that we have $m \geq 2^{1+14/g} \geq 91.7$, and so $\log_2 m - 2 \geq \ln m$ and hence we have $\frac{\ln m}{a} \leq g$.

We form the fractional solution \hat{x} by setting $\hat{x}_i = \frac{a}{2^{q-1}}$ for $i = 1, \dots, n$. This shows that the optimal fractional solution has value $\hat{T} \leq \frac{(2^q-1)a}{2^{q-1}} \leq 2a$.

Now consider some integral solution $x \in \mathbf{Z}_+^n$ with $\sum_i x_i = T$. We can write x as a sum of basis vectors, $\vec{x} = e_{y_1} + \dots + e_{y_T}$, where y_1, \dots, y_T are not necessarily distinct. Consider the quantity

$$V = \sum_k \sum_{1 \leq i_1 < \dots < i_{q-1} \leq T} [k \cdot y_{i_1} = \dots = k \cdot y_{i_{q-1}} = 0]$$

where $[]$ is the Iverson notation (which is one if $k \cdot y_{i_1} = \dots = k \cdot y_{i_{q-1}} = 0$ and zero otherwise).

We count V in two different ways. First, for any i_1, \dots, i_{q-1} , by linear algebra over $GF(2)$ there must exist at least one $k \neq 0$ which is orthogonal to all $y_{i_1}, \dots, y_{i_{q-1}}$. Hence we have $V \geq \binom{T}{q-1}$.

Second, for any k , there are at most $T - a$ choices of y_i which are orthogonal to k . Thus we have $V \leq (2^q - 1) \binom{T-a}{q-1}$. We have shown a lower bound on V and an upper bound on V . The lower bound on V must be at most the upper bound on V , or otherwise we would have a contradiction. Thus, a necessary condition for x to satisfy the covering constraints is that

$$\frac{\binom{T}{q-1}}{\binom{T-a}{q-1}(2^q - 1)} \leq 1 \tag{4}$$

We claim that (4) implies that $T > (q-1)(2/g + 1/4)$. As the LHS of (4) is a decreasing function of T , it suffices to show that (4) is violated for $T = (q-1)(2/g + 1/4)$. Rearranging some terms and recalling that $a = (q-1)/g$, we see that it suffices to show that

$$\frac{\binom{(q-1)(2/g+1/4)}{q-1}}{\binom{(q-1)(1/g+1/4)}{q-1}(2^q - 1)} > 1 \tag{5}$$

We use the bounds $2^q - 1 \leq 2^q$ and the bound on the factorial $\sqrt{2\pi r} r^{r+\frac{1}{2}} e^{-r} \leq r! \leq e r^{r+\frac{1}{2}} e^{-r}$, to obtain the following condition, which implies (5):

$$2^{-q}(4-3g)^{\frac{-3gq+5g+4q-4}{4g}}(8-3g)^{\frac{3gq-5g-8q+8}{4g}}(g+4)^{-\frac{(g+4)q+g-4}{4g}}(g+8)^{\frac{(g+8)q+g-8}{4g}} > \frac{e^2}{2\pi} \tag{6}$$

We can increase the RHS of (6) slightly to e to simplify the calculations, and take the logarithm to solve for q . This gives us the following condition, which implies (6):

$$q > 1 + \frac{2g(-2 + \ln(4 - 3g) - \ln(8 - 3g) - \ln(g + 4) + \ln(g + 8) - 2 \ln 2)}{4g \ln 2 - (4 - 3g) \ln(4 - 3g) + (8 - 3g) \ln(8 - 3g) + (4 + g) \ln(4 + g) - (8 + g) \ln(8 + g)} \quad (7)$$

The RHS of (7) is a function of g alone. Simple but tedious analysis (see Proposition A.5) shows that it is at most $14/g$.

But note that $q = \lfloor \log_2 m \rfloor \geq \log_2 m - 1$; thus, our bound on the size of m guarantees that indeed $q > 14/g$. So (7) \Rightarrow (6) \Rightarrow (5) $\Rightarrow T \geq (q - 1)(2/g + 1/4)$. The integrality gap is then given by

$$T/\hat{T} \geq \frac{(q - 1)(2/g + 1/4)}{2a} = \frac{2a + ag/4}{2a} = 1 + g/8$$

□

7 Multi-criteria Programs

One extension of the covering integer program framework is the presence of multiple linear objectives. Suppose now that instead of a single linear objective, we have multiple objectives $C_1 \cdot x, \dots, C_r \cdot x$. We also may have some over-all objective function D defined by the following:

$$D(x_1, \dots, x_n) = D(C_1 \cdot x, \dots, C_r \cdot x)$$

For example, we might have $D = \max_l C_l \cdot x$ or we might have $D = \sum_l (C_l \cdot x)^2$.

We note that the greedy algorithm, which is powerful for set cover, is not obviously useful in this case. However, depending on the precise form of the function D , it may be possible to solve the fractional relaxation to optimality. For example, if $D = \max_l C_l \cdot x$, then this amounts to a linear program of the form $\min t$ subject to $C_l \cdot x \leq t$.

For our purposes, the algorithm used to solve the fractional relaxation is not relevant. Suppose we are given some solution \hat{x} . We now want to find a solution x such that we have *simultaneously* $C_\ell \cdot x \approx C_\ell \cdot \hat{x}$ for all ℓ . Showing bounds on the expectations alone is not sufficient — it might be the case that $\mathbf{E}[C_\ell \cdot x] \leq \beta C_\ell \cdot \hat{x}$, but the random variables $C_1 \cdot x, \dots, C_r \cdot x$ are negatively correlated.

In [19], Srinivasan gave a construction which provided this type of simultaneous approximation guarantee. This algorithm was based on randomized rounding, which succeeded only with an exponentially small probability. Srinivasan also gave a derandomization of this process, leading to a somewhat efficient algorithm. This derandomization had a somewhat worsened approximation ratio compared to the single-criterion setting, roughly of the order $O(1 + \frac{\log(\Delta_0 + 1)}{a_{\min}})$, and running time of $O(n^{\log r})$. In particular, this was only polynomial if r was constant.

In this section, we will show that at the end of the ROUNDING algorithm, the values of $C_\ell \cdot x$ are concentrated around their means. This will establish that there is a good probability that we have $C_\ell \cdot x \approx \mathbf{E}[C_\ell \cdot x]$ for all $\ell = 1, \dots, r$. Thus, our algorithm automatically gives good approximation

ratios for multi-criteria problems; the ratios are essentially the same as for the single-criterion setting, and there is no extra computational burden.

We begin by showing that the values of x produced by the RELAXATION algorithm obey a type of negative correlation property. We will show this via a type of “witness” construction, similar to Lemma 2.1; however, instead of providing a witness for the event that $x_i = 1$, we will provide a witness for the event that simultaneously $x_{i_1} = \dots = x_{i_s} = 1$.

This proof is based on induction similar to Lemma 2.1. Suppose we are given any set $I \subseteq [n]$, any integers $J_1, \dots, J_m \geq 0$, and an array of sets $Z = \langle Z_{j,k} \mid k = 1, \dots, m, j = 1, \dots, J_k \rangle$. We then define the event $\mathcal{E}(I, J, Z)$ to be the following:

1. For each $k = 1, \dots, m$, the first J_k resampled sets for constraint k are respectively $Z_{k,1}, \dots, Z_{k,J_k}$
2. Each $i \in I$ turns at 0 or some (k, j) where $1 \leq j \leq J_k$.

We similarly define the event $\mathcal{E}(I, J, Z, v)$ for any $v \in \{0, 1\}^n$ to be that the event $\mathcal{E}(I, J, Z)$ occurs, if we start the RELAXATION algorithm by setting $x = v$ (instead of drawing x as independent Bernoulli- p_i), and the event $\mathcal{E}(T, I, J, Z, v)$ to be the event that $\mathcal{E}(I, J, Z, v)$ occurs *and* the relaxation algorithm terminates in less than T resamplings.

Given any integers J_1, \dots, J_m , we define $\text{prefix}(J)$ to be the set of all pairs (k, j) where $1 \leq j \leq J_k$.

Proposition 7.1. *Suppose that $x_i \in [0, 1/\alpha]^n$. Let $v \in \{0, 1\}^n$, $I \subseteq [n]$, and J, Z be given.*

Then

$$P(\mathcal{E}(I, J, Z)) \leq \prod_{i \in I} p_i \prod_{(k,j) \in \text{prefix}(J)} f_k(Z_{k,j})$$

Proof. Define

$$D = \bigcup_{(k,j) \in \text{prefix}(J)} Z_{j,k}$$

We prove by induction on T that for any $T \geq 0$ we have

$$P(\mathcal{E}(T, I, J, Z, v)) \leq \prod_{i \in I \cap D} p_i \frac{\prod_{(k,j) \in \text{prefix}(J)} f_k(Z_{k,j})}{\prod_{i \in D} (1 - p_i)}$$

A few details of the proof which are identical to Lemma 2.1 are omitted for clarity.

Let k be minimal such that $A_k \cdot x < a_k$. If $J_l \geq 1$ for any $l < k$ then the event $\mathcal{E}(T, I, J, Z, v)$ is impossible and we are done. If $J_k = 0$, then $\mathcal{E}(T, I, J, Z, v)$ is equivalent to $\mathcal{E}(T - 1, I, J, Z, x')$ where x' is the value of the variables after a resampling; for this we use the induction hypothesis and we are done.

So suppose $J_k \geq 1$. Define $D' = \bigcup_{\substack{(j,l) \in \text{prefix}(J) \\ (j,l) \neq (1,k)}} Z_{j,l}$. Then the following are necessary events to have $\mathcal{E}(T - 1, I, J, Z, x')$:

- (A1) We select $Z_{k,1}$ as the resampled set for constraint k
- (A2) The event $\mathcal{E}(T-1, I', J', Z', x')$ occurs, where x' is the value of the variables after resampling, where $I' = I \cap D'$, and J', Z' are derived by setting $J'_k = J_k - 1$ and by $Z'_{k,1}, \dots, Z'_{k,J_k-1} = Z_{k,2}, \dots, Z'_{k,J_K}$ (and all other entries remain the same)
- (A3) For any $i \in (Z_{k,1} - D') \cap I$ we resample $x_i = 1$
- (A4) For any $i \in Z_{k,1} \cap D'$ we resample $x_i = 0$

The rationale for (A3) is that we require $i \in I$ to turn at some $(j, l) \in \text{prefix}(J)$, and in addition $Z_{j,l}$ is the j^{th} resampled set for constraint l . This would imply that $i \in Z_{j,l}$. However, there is only one such (j, l) , namely $(j, l) = (1, k)$. Thus, we are requiring i to become resampled to $x_i = 1$.

The rationale for (A4) is the same as in Lemma 2.1: if we resample $x_i = 1$, then x_i can never be resampled again. In particular, we cannot have i in any future resampled set. Thus if $x'_i = 1$ but $i \in Z_{k,1} \cap D'$, then the event (A2) is impossible.

As in Lemma 2.1, the event (A1) has probability $\leq (1 - \sigma)^{-a_k} \prod_{i \in [n]} (1 - A_{ki} \sigma) \prod_{i \in Z_{k,1}} \frac{A_{ki} \sigma}{1 - A_{ki} \sigma}$.

Event (A3), conditional on (A1), has probability $\prod_{i \in (Z_{k,1} - D') \cap I} p_i$.

Event (A4), conditional on (A1), (A3), has probability $\prod_{i \in Z_{k,1} \cap D'} 1 - p_i$.

By induction hypothesis, event (A2), conditional on (A1), (A3), (A4), has probability

$$P((A2)) \leq \prod_{i \in I' - D'} p_i \times \prod_{i \in D'} (1 - p_i) \times \prod_{(j,l) \in \text{prefix}(J')} f_k(Z_{k,l})$$

Multiplying these probabilities, after some rearrangement, gives us the desired bound on $P(\mathcal{E}(T, I, J, Z, v))$, thus completing the induction.

Next, as in Lemma 2.1, we immediately obtain also

$$P(\mathcal{E}(I, J, Z, v)) = \lim_{T \rightarrow \infty} P(\mathcal{E}(T, I, J, Z, v)) \leq \prod_{i \in I \cap D} p_i \times \frac{\prod_{(k,j) \in \text{prefix}(J)} f_k(Z_{k,j})}{\prod_{i \in D} (1 - p_i)}$$

Finally, to obtain a bound on $P(\mathcal{E}(I, J, Z))$, we observe that if $i \in D$, then x_i must be equal to zero during the initial sampling. Also, if $i \in I - D$, then x_i must be equal to one during the initial sampling. This has probability $\prod_{i \in I - D} p_i \prod_{i \in D} (1 - p_i)$. Conditional on this event, we have $P(\mathcal{E}(I, J, Z, x)) \leq \prod_{i \in I \cap D} p_i \times \frac{\prod_{(k,j) \in \text{prefix}(J)} f_k(Z_{k,j})}{\prod_{i \in D} (1 - p_i)}$. Thus, multiplying the probabilities together, gives us

$$P(\mathcal{E}(I, J, Z)) \leq \prod_{i \in I} p_i \prod_{(k,j) \in \text{prefix}(J)} f_k(Z_{k,j})$$

as desired. □

Proposition 7.2. *Let $R \subseteq [n]$. Suppose that at the end of the RELAXATION algorithm we have $x_i = 1$ for all $i \in R$.*

Then there is a set $R' \subseteq R$ and an injective function $h : R' \rightarrow [m]$, as well as non-negative integers J_1, \dots, J_m and sets $Z_{k,j}$ for $j = 1, \dots, J_k$, which satisfy the following properties:

- (D1) *For all $i \in R'$ we have $J_{h(i)} \geq 1$ and $i \in Z_{h(i), J_{h(i)}}$*
- (D2) *For all $k \notin h(R')$ we have $J_k = 0$*
- (D3) *Each $i \in R$ turns at either 0 or at some (k, j) for $k \leq J_k$*

Proof. Let $S_0 \subseteq R$ denote the set of variables $i \in R$ which turn at 0. For each $k = 1, \dots, m$ let $S_k \subseteq R$ denote the variables $i \in R$ which turn at constraint k , where each $i \in S_k$ turns at (k, L_i) . Observe that S_0, S_1, \dots, S_m form a partition of R .

Now for each $k = 1, \dots, m$ we define:

$$J_k = \max_{i \in S_k} L_i$$

We form the set R' by selecting, for each $k \in [m]$ with $S_k \neq \emptyset$, exactly one $i \in S_k$ with $L_i = J_k$ (there may be more than one; in which case we select i arbitrarily). We define f by mapping this $i \in S_k$ to k .

Note that we must have $i \in Z_{h(i), J_{h(i)}}$, as we are assuming that $L_i = J_k$ where $k = h(i)$.

Also, each $i \in S_k$ must turn at (k, L_i) and $L_i \leq J_k$, thus (D3) is satisfied. □

Theorem 7.3. *Suppose $x \in [0, 1/\alpha]^n$ and $\alpha > \frac{-\ln(1-\sigma)}{\sigma}$. For any $R \subseteq [n]$, the probability that $x_i = 1$ for all $i \in R$ is at most*

$$P\left(\bigwedge_{i \in R} x_i = 1\right) \leq \prod_{i \in R} \rho_i$$

where, for each $i \in [n]$, we define

$$\rho_i = \alpha \hat{x}_i \left(1 + \sigma \sum_k \frac{A_{ki}}{(1-\sigma)^{a_k} e^{\sigma \alpha A_k \cdot \hat{x}} - 1} \right)$$

Proof. By Proposition 7.2, there must exist $R', h, Z_{k,j}, J$ satisfying (D1), (D2), (D3). By Lemma 7.1, for any Z, J satisfying (D1), (D2) the probability of satisfying (D3) is at most $\prod_{i \in R} p_i \prod_{(k,j) \in \text{prefix}(J)} f_k(Z_{k,j})$. Taking a union bound over all such $J, Z_{k,j}$ we have:

$$P\left(\bigwedge_{i \in R} x_i = 1\right) \leq \sum_{\substack{R', h, Z, J \\ \text{satisfying (D1), (D2)}}} \prod_{i \in R} p_i \prod_{(k,j) \in \text{prefix}(J)} f_k(Z_{k,j}) \quad (8)$$

We must enumerate over all R', h, Z, J satisfying (D1), (D2). Suppose now that R' and h are fixed. To simplify the notation, let us suppose wlg that $R' = \{1, \dots, r\}$. We now consider the following process to enumerate over Z, J :

1. We select any vector of integers $J' \in \mathbf{Z}_+^r$, and sets $Z'_{i,j}$ where $j \leq J'_i$.
2. For each $i \in R'$, we select a set $W_i \subseteq [n]$ with $i \in W_i$.
3. We define J by $J_{h(i)} = J'_i + 1$ for $i = 1, \dots, r$, and all other value of J are equal to zero. Also for $j = 1, \dots, J'_i$ we set $Z_{h(i),j} = Z'_{i,j}$ and finally $Z_{h(i),J'_i+1} = W_i$.

Now observe that for a fixed R', h this process enumerates every Z, J satisfying (D1), (D2) exactly once. Furthermore, for any J', Z', W , we have

$$\prod_{(k,j) \in \text{prefix}(J)} f_k(Z_{k,j}) = \sum_{W_1 \ni 1, \dots, W_r \ni r} \prod_{i \in R'} f_{h(i)}(W_i) \prod_{i=1}^r \prod_{j=1}^{J'_i} f_k(Z_{j,k})$$

Thus, summing over possible values for Z', J', W we have:

$$\begin{aligned} \sum_{\substack{Z, J \\ \text{satisfying (D1), (D2)}}} \prod_{(k,j) \in \text{prefix}(J)} f_k(Z_{k,j}) &= \prod_{i=1}^r \left(\sum_{W \subseteq [n], W \ni i} f_{h(i)}(W_i) \sum_{j' \geq 0} \sum_{Z_{h(i),1}, \dots, Z_{h(i),j'}} \prod_{l=1}^{j'} f_k(Z_{h(i),l}) \right) \\ &\leq \prod_{i=1}^r s_{h(i)} A_{h(i),i} \sigma \times \sum_{j' \geq 0} (s_{h(i)})^{j'} \quad \text{by Propositions 2.2, 2.3} \\ &= \prod_{i \in R'} \frac{s_{h(i)} A_{h(i),i} \sigma}{1 - s_{h(i)}} \end{aligned}$$

Thus, now we may sum over $R' \subseteq R$ and injective $h : R' \rightarrow [m]$ as:

$$\begin{aligned} \sum_{\substack{R', h, Z, J \\ \text{satisfying (D1), (D2)}}} \prod_{i \in R} p_i \prod_{(k,j) \in \text{prefix}(J)} f_k(Z_{k,j}) &\leq \prod_{i \in R} p_i \sum_{\substack{R' \subseteq R \\ \text{injective } h : R' \rightarrow [m]}} \prod_{i \in R'} \frac{s_{h(i)} A_{h(i),i} \sigma}{1 - s_{h(i)}} \\ &\leq \prod_{i \in R} p_i \sum_{\substack{R' \subseteq R \\ h : R' \rightarrow [m]}} \prod_{i \in R'} \frac{s_{h(i)} A_{h(i),i} \sigma}{1 - s_{h(i)}} \\ &= \prod_{i \in R} p_i \sum_{R' \subseteq R} \prod_{i \in R'} \sum_{k=1}^m \frac{s_k A_{k,i} \sigma}{1 - s_k} \\ &= \prod_{i \in R} p_i \left(1 + \sum_{k=1}^m \frac{s_k A_{k,i} \sigma}{1 - s_k} \right) = \prod_{i \in R} \rho_i \end{aligned}$$

□

We can now show a concentration phenomenon for $C \cdot x$. In order to obtain the simplest such bounds, we can make an assumption that the entries of C are in the range $[0, 1]$. In this case, we can use the Chernoff upper-tail function to give estimates for the concentration of $C \cdot x$.

Definition 7.4 (The Chernoff upper-tail). For $t \geq \mu$ with $\delta = \delta(\mu, t) = t/\mu - 1 \geq 0$, the Chernoff upper-tail bound is defined as

$$\text{Chernoff-U}(\mu, t) = \left(\frac{e^\delta}{(1+\delta)^{1+\delta}} \right)^\mu \quad (9)$$

That is to say $\text{Chernoff-U}(\mu, t)$ is the Chernoff bound that a sum of $[0, 1]$ -bounded and independent random variables with mean μ will be above t .

Corollary 7.5. Suppose that all entries of C_l are in the interval $[0, 1]$ and that $\hat{x} \in [0, 1/\alpha]^n$ and $\alpha > \frac{-\ln(1-\sigma)}{\sigma}$. Then, after running the RELAXATION algorithm, the probability of the event $C_l \cdot x > t$ is at most $\text{Chernoff-U}(C_l \cdot \rho, t)$.

Proof. The value of $C_l \cdot x$ is a sum of random variables $C_{li}x_i$ which are in the range $[0, 1]$. These random variables obey a negative-correlation property as shown in Theorem 7.3. This implies that they obey the same upper-tail Chernoff bounds as would a sum of random variables X_i which are independent and satisfy $\mathbf{E}[X_i] = \rho_i$. \square

We next need to show concentration for the ROUNDING algorithm.

Theorem 7.6. Suppose that all entries of C_l are in $[0, 1]$. Then, after the ROUNDING algorithm, the probability of the event $C_l \cdot x > t$ is at most $\text{Chernoff-U}(C_l \cdot \rho, t)$.

Proof. Let $v_i, G_i, \hat{x}'_i, a'_k, x'$ be the variables which occur during the ROUNDING algorithm. We have

$$\begin{aligned} P(C_l \cdot x > t) &= P(C_l \cdot (v\theta + G + x') > t) \\ &= P(C_l \cdot x' > t - C_l \cdot (v\theta + G)) \\ &\leq \text{Chernoff-U}(C_l \cdot \rho', t - C_l \cdot (v\theta + G)) \end{aligned}$$

Thus, we have

$$P(C_l \cdot x > t) \leq \text{Chernoff-U}\left(\alpha \sum_i C_{li} \hat{x}'_i, \left(1 + \sigma \sum_k \frac{A_{ki}}{(1-\sigma)^{a'_k} e^{\sigma \alpha A_k \cdot \hat{x}'} - 1}\right), t - C_l \cdot (v\theta + G)\right) \quad (10)$$

By Proposition A.3, $\text{Chernoff-U}(\mu, t)$ is always an increasing function of μ . So we can show an upper bound for this expression by giving an upper bound for the μ term in the (10). We first apply Propositions 3.2, 3.4 which give:

$$x'_i \left(1 + \sigma \sum_k \frac{A_{ki}}{(1-\sigma)^{a'_k} e^{\sigma \alpha A_k \cdot \hat{x}'} - 1} \right) \leq (\hat{x}_i - v_i \theta - G_i/\alpha) T$$

where we define

$$T = 1 + \sigma \sum_k \frac{A_{ki}}{(1-\sigma)^{a_k} e^{\sigma \alpha A_k} - 1}$$

Substituting this upper bound into (10) yields:

$$P(C_l \cdot x > t) \leq \text{Chernoff-U}\left(\alpha \sum_i C_{li} (\hat{x}_i - v_i \theta - G_i/\alpha) T, t - C_l \cdot (v\theta + G)\right)$$

$$\begin{aligned}
&\leq \text{Chernoff-U}\left(\sum_i C_{li}(\rho_i - (v_i\alpha\theta + G_i)), t - C_l \cdot (v\theta + G)\right) \\
&\leq \text{Chernoff-U}\left((C_l \cdot \rho) - (C_l \cdot (v\theta + G)), t - (C_l \cdot (v\theta + G))\right) \\
&\leq \text{Chernoff-U}(C_l \cdot \rho, t) \quad \text{by Proposition A.4}
\end{aligned}$$

□

In the column-sparsity setting, we obtain the following result which extends Theorem 5.1:

Corollary 7.7. *Suppose we are given a covering system as well as a fractional solution \hat{x} . Let $\gamma = \frac{\log(\Delta_1+1)}{a_{\min}}$. Suppose that the entries of C_l are in $[0, 1]$. Then, with an appropriate choice of σ, α we may run the ROUNDING algorithm in expected time $O(mn)$ to obtain a solution $x \in \mathbf{Z}_+^n$ such that*

$$P(C_l \cdot x > t) \leq \text{Chernoff-U}(\beta C_l \cdot \hat{x}, t)$$

for $\beta = 1 + \gamma + 4\sqrt{\gamma}$.

If one wishes to ensure also that $x_i \leq \lceil \hat{x}_i(1 + \epsilon) \rceil$ for $\epsilon \in (0, 1)$, then one can obtain a similar result with an approximation factor $\beta = 1 + 4\sqrt{\gamma} + 4\gamma/\epsilon$.

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A Some technical lemmas

Proposition A.1. *Given set S , $x_i \in [0, 1]$, and $a \in (0, 1)$, we have*

$$\prod_{i \in S} (1 - ax_i)^{-1} \leq (1 - a)^{-\sum_{i \in S} x_i}$$

Proof. Using the concavity of the function $\prod_{i \in S} (1 - ax_i)^{-1}$ we can show that, for a fixed value of $s = \sum_{i \in S} x_i$, it attains a maximum occurs when at most one x_i is fractional. So suppose $s = z + r$ where $z \in \mathbf{Z}_+$ and $r \in (0, 1)$. Then we have

$$\prod_{i \in S} (1 - ax_i)^{-1} = (1 - a)^{-z} (1 - ra)^{-1} \leq (1 - a)^{-z} (1 - a)^{-r} = (1 - a)^{-\sum_{i \in S} x_i}.$$

□

Proposition A.2. For any $\gamma > 0$, define

$$f(x) = \frac{x}{(x+1)^{\frac{2\sqrt{\gamma}+\gamma-2\ln(1+\sqrt{\gamma})}{\gamma}} - 1}$$

Then $f(x)$ is a decreasing function of x for $x > 0$.

Proof. At $x = 0$, both numerator and denominator are equal to zero. So it suffices to show that the denominator grows faster than the numerator, that is, that the derivative of the denominator is always ≥ 1 . We compute the derivative of the denominator:

$$\begin{aligned} R &= \frac{(\gamma + 2\sqrt{\gamma} - 2\ln(\sqrt{\gamma} + 1))(x+1)^{\frac{2\sqrt{\gamma}-2\ln(1+\sqrt{\gamma})}{\gamma}}}{\gamma} \\ &\geq \frac{(\gamma + 0)(x+1)^{\frac{0}{\gamma}}}{\gamma} \quad \text{as } y \geq \ln(1+y) \text{ for } y > 0 \\ &= 1 \quad \text{as desired} \end{aligned}$$

□

Proposition A.3. For any $0 \leq \mu \leq \mu' \leq t$ we have $\text{Chernoff-U}(\mu, t) \leq \text{Chernoff-U}(\mu', t)$.

Proof. Compute the partial derivative of $\text{Chernoff-U}(\mu, t)$ with respect to μ . □

Proposition A.4. For any $0 \leq \mu \leq t$ and any $r \leq \mu$, we have $\text{Chernoff-U}(\mu, t) \leq \text{Chernoff-U}(\mu - r, t - r)$.

Proof. Compute the directional derivative of $\text{Chernoff-U}(\mu, t)$ along the unit vector $\hat{u} = (1, 1)$. □

Proposition A.5. Let $f_1(g) = 2g(-2 + \ln(4 - 3g) - \ln(8 - 3g) - \ln(g + 4) + \ln(g + 8) - 2\ln 2)$ and let $f_2(g) = g\ln 2 - (4 - 3g)\ln(4 - 3g) + (8 - 3g)\ln(8 - 3g) + (4 + g)\ln(4 + g) - (8 + g)\ln(8 + g)$. For any $g \in (0, 1)$ we have

$$14/g > 1 + \frac{f_1(g)}{f_2(g)} \tag{11}$$

Proof. Let us first consider the denominator $f_2(g)$. Note that $f_2''(g)$ is a rational function, and simple algebra shows that its only root is at $g = -16/9$. As $f_2''(0) = -1$, this implies that $f_2''(g) < 0$ for all $g \in (0, 1)$. Thus, $f_2'(g)$ is decreasing in this range. As $f_2'(0) = 0$, this implies that $f_2'(g) < 0$ for $g \in (0, 1)$. As $f_2(0) = 0$, this further implies that $f_2(g) < 0$ for $g \in (0, 1)$.

We may thus cross-multiply (11), taking into account the fact that the denominator is negative. Thus to show (11) suffices to show that $h(g) < 0$, where we define

$$h(g) = (-5g^2 + 46g - 56) \ln(4 - 3g) + (5g^2 - 50g + 112) \ln(8 - 3g) \\ + (g^2 + 10g + 56) \ln(g + 4) + (-g^2 - 6g - 112) \ln(g + 8) + 4g^2 + 56g \ln 2$$

Simple calculus shows that $h'''(g)$ is a rational function of g , and it has no roots in the range $(0, 1)$. As $h'''(0) = -75/8$, this implies that $h'''(g) < 0$ for all $g \in (0, 1)$. As $h''(0) = -0.454$, this implies that $h''(g) < 0$ for all $g \in (0, 1)$. As $h'(0) = 0$, this implies that $h'(g) < 0$ for all $g \in (0, 1)$. As $h(0) = 0$, this implies that $h(g) < 0$ for all $g \in (0, 1)$. \square